Subgraphs and Well-Quasi-Ordering

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ABSTRACT

Let $\mathcal{G}$ be a class of graphs and let $\leq$ be the subgraph or the induced subgraph relation. We call $\mathcal{G}$ an ideal (with respect to $\leq$) if $G \leq G' \in \mathcal{G}$ implies that $G \in \mathcal{G}$. In this paper, we study the ideals that are well-quasi-ordered by $\leq$. The following are our main results. If $\leq$ is the subgraph relation, we characterize the well-quasi-ordered ideals in terms of excluding subgraphs. If $\leq$ is the induced subgraph relation, we present three well-quasi-ordered ideals. We also construct examples to disprove some of the possible generalizations of our results. The connections between some of our results and digraphs are considered in this paper too. ©1992 John Wiley & Sons, Inc.

1. INTRODUCTION

A binary relation $\leq$ defined on a set $Q$ is a quasi-ordering if $\leq$ is reflexive and transitive. A sequence $q_1, q_2, \ldots$ of members of $Q$ is called a good sequence (with respect to $\leq$) if there exist $i < j$ such that $q_i \leq q_j$. It is a bad sequence if otherwise. We call $(Q, \leq)$ a well-quasi-ordering (or a wqo) if there is no infinite bad sequence. An ideal of $Q$ (with respect to $\leq$) is a subset $Q'$ of $Q$ such that $q \leq q' \in Q'$ implies that $q \in Q'$. Clearly, if $(Q, \leq)$ is a wqo and if $Q'$ is an ideal of $Q$, then $Q'$ can be characterized by a Kuratowski type theorem. Namely, $Q'$ can be characterized by excluding finitely many members of $Q$.

Let $Q$ be the class of all finite simple graphs and let $\leq$ be the subgraph or induced subgraph relation. Then $(Q, \leq)$ is not a wqo as shown by the bad sequence $C_3, C_4, \ldots$ of circuits. However, if $Q$ is restricted to some smaller class of graphs, $(Q, \leq)$ could be a wqo even if $\leq$ is the induced subgraph

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Throughout this paper, ≤ is reserved for the induced subgraph relation. As usual, \( P_n \), \( C_n \), and \( K_n \) are respectively the path, circuit, and clique on \( n \) vertices. For \( n = 1, 2, \ldots \), let \( \mathcal{P}_n \) be the class of graphs without \( P_n \) subgraph (notice that this is a stronger restriction than "without \( P_n \) induced subgraph"). We shall prove in Section 2 that \( (\mathcal{P}_n, \leq) \) is a wqo. In fact, we are able to prove an even stronger result. This stronger result is used to show that if \( Q \) is a graph ideal (with respect to the subgraph relation), then \( Q \) is well-quasi-ordered by the subgraph relation if and only if it is well-quasi-ordered by the induced subgraph relation. We characterize these graph ideals in terms of excluding subgraphs. In the same section, the connections between our results and digraphs are also considered. In Section 4, we restrict ourselves to bipartite graphs. We present two classes of bipartite graphs that are well-quasi-ordered by \( \leq \). We also construct examples in Section 3 to disprove some of the possible generalizations of our results.

2. EXCLUDING LONG PATHS

Let \( (Q, \leq) \) be a quasi-ordering and let \( G \) be a graph. A \( Q \)-labeling of \( G \) is a mapping \( f \) from \( V(G) \) to \( Q \). We call the pair \( (G, f) \) \( Q \)-labeled graph. If \( \mathcal{F} \) is a class of graphs, we denote by \( \mathcal{F}(Q) \) the class of \( Q \)-labeled graphs \( (G, f) \) such that \( G \in \mathcal{F} \). For any two members \( (G, f) \) and \( (G', f') \) of \( \mathcal{F}(Q) \), we define \( (G', f') \leq (G, f) \) if \( G \leq G' \) and \( f(v) \leq f'((\sigma(v)) \) for all \( v \in V(G) \), where \( \sigma \) is an isomorphism from \( G \) to an induced subgraph of \( G' \). The main result of this section is the following.

**Theorem 2.1.** \( (\mathcal{P}_n(Q), \leq) \) is a wqo provided \( (Q, \leq) \) is.

As a consequence, we have

**Theorem 2.2.** \( (\mathcal{P}_n, \leq) \) is a wqo.

To prove Theorem 2.1, we need a few lemmas. We first define the type of a graph as follows. A single vertex has type 1, and inductively for \( n > 1 \), a graph \( G \) has type at most \( n \) if for some vertex \( v \in V(G) \), every connected component of \( G \setminus v \) has type at most \( n - 1 \). It is not difficult to show [5] that

**Lemma 2.3.** Every graph in \( \mathcal{P}_n \) has type at most \( n \).

For a quasi-ordering \( (Q, \leq) \), let \( Q^* \) be the set of all finite sequences of members of \( Q \). Suppose \( (q_1, \ldots, q_i) \) and \( (q'_1, \ldots, q'_i) \) are members of \( Q^* \). We define \( (q_1, \ldots, q_i) \leq^* (q'_1, \ldots, q'_i) \) if there exist indices \( 1 \leq i_1 < \cdots < i_t \leq i \) such that \( q_i \leq q_{i_1}, \ldots, q_{i_t} \leq q'_{i_t} \). The following result is due to Higman [4]:
Lemma 2.4. \((Q^*, \leq^*)\) is a wqo provided \((Q, \leq)\) is.

Let \(\mathcal{F}_n\) be the class of graphs of type at most \(n\). From Lemma 2.3 it is clear that to prove Theorem 2.1, we only need to show

Lemma 2.5. \((\mathcal{F}_n(Q), \preceq)\) is a wqo provided \((Q, \leq)\) is.

Proof. We prove the lemma by induction on \(n\). Since the only graph of type one is \(K_1\), it follows that \((\mathcal{F}_1(Q), \preceq)\) is nothing but \((Q, \leq)\). Therefore the result is true for \(n = 1\) and for every wqo \((Q, \leq)\).

Suppose now \(n > 1\). Let \((G_1, f_1), (G_2, f_2), \ldots\) be an infinite sequence of members of \(\mathcal{F}_n(Q)\). We need to show that there exist indices \(i < j\) such that \((G_i, f_i) \preceq_i (G_j, f_j)\). For each \(i = 1, 2, \ldots\), let \(v_i\) be a vertex of \(G_i\) such that every connected component of \(G_i \setminus v_i\) has type at most \(n - 1\). By taking an infinite subsequence if necessary, we may assume that \(f_1(v_1) \leq f_2(v_2) \leq \cdots\) (we can do this since \((Q, \leq)\) is a wqo). Let \(G_{11}, \ldots, G_{1k_1}\) be the connected components of \(G_1 \setminus v_1\), and let \(f_{11}, \ldots, f_{1k_1}\) be their \(Q'\)-labelings respectively, where \(Q' = Q \times \{0, 1\}\) and \(f_{0j}(v) = (f_j(v), e_j(v))\) (\(v \in V(G_j)\)), such that \(e_j(v) = 1\) if and only if \(v\) is adjacent to \(v_j\) in \(G_j\). Clearly, all the labeled graphs \((G_{ij}, f_{ij})\) are members of \(\mathcal{F}_{n-1}(Q')\). For any \((q, e), (q', e') \in Q'\), we define \((q, e) \preceq' (q', e')\) if \(q \leq q'\) and \(e = e'\). It is not difficult to see that \((Q', \preceq')\) is a wqo provided \((Q, \leq)\) is. Thus from the hypothesis of our induction we deduce that \(\mathcal{F}_{n-1}(Q')\) is a wqo. Now it follows from Lemma 2.4 that there exist indices \(i < j\) such that \((G_{11}, f_{11}), \ldots, (G_{1k_1}, f_{1k_1}) \preceq^* (G_{j1}, f_{j1}), \ldots, (G_{jk_j}, f_{jk_j})\).

We claim that \((G_i, f_i) \preceq_i (G_j, f_j)\). Let \(1 \leq j_1 < \cdots < j_s \leq k_j\) such that \((G_{i1}, f_{i1}) \preceq_l (G_{j1}, f_{j1}), \ldots, (G_{is}, f_{is}) \preceq_l (G_{js}, f_{js})\), where \(s = k_j\). From the definition of \(\preceq_i\), we deduce that for each \(k = 1, \ldots, s\), there exists an induced subgraph \(G'_{ik}\) of \(G_{jk}\), and an isomorphism \(\sigma_k\) from \(G_{ik}\) to \(G'_{ik}\), such that \(f_{ik}(v) = \sigma_k(f_{ik}(v))\) for all \(v \in V(G_{ik})\). Let \(\sigma\) be the mapping from \(V(G_i)\) to \(V(G_j)\) such that \(\sigma(v_i) = v_j\) and for every other vertex \(v \in V(G_i)\), \(\sigma(v) = \sigma(v)\) if \(v \in V(G_i)\). Clearly, \(\sigma\) is an isomorphism from \(G_i\) to an induced subgraph of \(G_j\) and \(f_i(v) \leq f_j(\sigma(v))\) for all \(v \in V(G_i)\). Thus \((G_i, f_i) \preceq_i (G_j, f_j)\) as required.

Remark. (1) It looks like Lemma 2.5 is stronger than Theorem 2.1 because of Lemma 2.3. But the fact is that they are equivalent. Readers are invited to prove the equivalence by showing that \(\mathcal{F}_n \subseteq P_n\) for all positive integers \(n\).

(2) Let \(\mathcal{F}\) be a class of graphs. We shall denote by \(\mathcal{D}(\mathcal{F})\) the class of all digraphs \(D\) such that the underlying graph of \(\mathcal{D}\) belongs to \(\mathcal{F}\). It is not difficult to see that a minor modification of our proof above proves that \((\mathcal{D}(P_n), \preceq_d)\) is a wqo, where the induced subdigraph relation \(\preceq_d\) is defined in the natural way.

Let us denote in the rest of this section that \(G_1 \subseteq G_2\) if \(G_1\) is a subgraph of \(G_2\), and \(D_1 \subseteq D_2\) if \(D_1\) is a subdigraph of \(D_2\).
Theorem 2.6. Let \( \mathcal{F} \) be an ideal of graphs with respect to \( \subseteq \). Then the following are equivalent:

(i) \( (\mathcal{F}, \leq_d) \) is a wqo;
(ii) \( (\mathcal{F}, \subseteq_d) \) is a wqo;
(iii) \( \mathcal{F} \) is contained in \( \mathcal{P}_n \) for some positive integer \( n \).

Proof. The implications (iii) \( \Rightarrow \) (i) and (i) \( \Rightarrow \) (ii) are clear. To prove (ii) \( \Rightarrow \) (iii), we only need to exhibit a bad sequence of digraphs (with respect to \( \subseteq_d \)) \( D_4, D_5, \ldots \) such that their underlying graphs are \( P_4, P_5, \ldots \), respectively. Let \( D_n \) \((n = 4, 5, \ldots)\) be defined on \( \{1, \ldots, n\} \) such that the arcs of \( D_n \) are \((1, 2), (3, 2), \ldots, (i, i-1), \ldots, (n-1, n-2), (n-1, n)\). Then it is clear that \( D_4, D_5, \ldots \) is a sequence satisfying the requirements.

In the rest of this section, we are going to characterize the ideals \( \mathcal{F} \) of graphs (with respect to \( \subseteq \)), such that \( (\mathcal{F}, \subseteq) \) is a wqo. We first observe that \( C_3, C_4, \ldots \) is a bad sequence. Another such bad sequence is \( F_1, F_2, \ldots \), where for each \( i = 1, 2, \ldots \),

\[
V(F_i) = \{y_1, y_2, z_1, z_2, x_1, \ldots, x_i\}
\]

and

\[
E(F_i) = \{y_1 x_1, y_2 x_1, z_1 x_i, z_2 x_i, x_1 x_2, x_2 x_3, \ldots, x_{i-1} x_i\}.
\]

Theorem 2.7. Let \( \mathcal{F} \) be an ideal of graphs with respect to \( \subseteq \). Then the following are equivalent:

(i) \( (\mathcal{F}, \subseteq) \) is a wqo;
(ii) \( (\mathcal{F}, \leq) \) is a wqo;
(iii) \( \mathcal{F} \) contains only finitely many graphs \( C_n \) and \( F_n \).

Proof. The implications (ii) \( \Rightarrow \) (i) and (i) \( \Rightarrow \) (iii) are clear. To prove (iii) \( \Rightarrow \) (ii), we assume that there exists a positive integer \( N \) such that \( C_n, F_n \notin \mathcal{F} \) for all \( n \geq N \). It follows that no graph in \( \mathcal{F} \) has a subgraph \( C_n \) or \( F_n \) for \( n \geq N \).

Let \( G \) be a graph. As usual, the degree of a vertex \( v \in V(G) \), denoted by \( d(v) \), is the number of edges incident with \( v \). An edge \( e = xy \) of \( G \) is called a bridge if (1) \( G \setminus e \) has one more connected components than \( G \) has; (2) the component of \( G \setminus e \) containing \( x \) is a path on at least two vertices such that \( x \) is an end of this path; and (3) the degree of \( y \) in \( G \) is at least three. It is clear that the degree of \( x \) in \( G \) is exactly two. Let \( Q \) be the set of positive integers. We define the condensation of a graph \( G \) to be a \( Q \)-labeled graph \((c(G), f)\) as follows. If \( G \) is disconnected, then the condensation of \( G \) is the disjoint union of the condensations of its connected components. If \( G \) is a path \( P_k \), then \( c(G) = K_1 \) and \( f(u) = k \) where \( \{u\} = V(K_1) \). If \( G \) is connected
and it is not a path, we assume that \( e_1 = x_1 y_1, \ldots, e_k = x_k y_k \) are all the bridges of \( G \), where \( k \geq 0 \). We also assume that \( e'_1, \ldots, e'_k \) are the other edges incident with \( x_1, \ldots, x_k \), respectively. Then \( c(G) \) is the connected component of \( G - \{ e'_1, \ldots, e'_k \} \) that contains \( x_1, \ldots, x_k \). For each \( i = 1, \ldots, k \), \( f(x_i) \) is defined to be \( n \) if the connected component of \( G - e_i \) that contains \( x_i \) is \( P_n \), and for every other vertex \( v \) of \( c(G) \), \( f(v) \) is defined to be 1. It is clear that for any two graphs \( G \) and \( G' \), \( (c(G), f) \preceq (c(G'), f') \) implies \( G \preceq G' \). Let \( \mathcal{C} \) be the class of graphs \( c(G) \) for all \( G \in \mathcal{F} \). Then our result follows if we can prove that \( \langle \mathcal{C}(Q), \preceq \rangle \) is a wqo. By Theorem 2.1, it is enough for us to prove that \( \mathcal{C} \) is a subset of \( \mathcal{P}_{3N} \).

Let \( G \in \mathcal{C} \). We need to show that \( G \in \mathcal{P}_{3N} \). Without loss of generality, we may assume that \( G \) is connected and \( |V(G)| \geq 3 \). We first observe, from the definition of \( \mathcal{C} \), that for every edge \( xy \) of \( G \), \( d(x) + d(y) \geq 4 \). Now suppose that \( G \) has a path \( P \) on \( k \geq 3N \) vertices \( x_1, \ldots, x_k \) such that \( E(P) = \{ x_1 x_2, x_2 x_3, \ldots, x_{k-1} x_k \} \). We chose \( P \) with \( k \) maximum. Since \( d(x_1) + d(x_2) \geq 4 \), there exists an edge \( e \in E(G) - E(P) \) such that \( e \) is incident with either \( x_1 \) or \( x_2 \). Similarly, there is an edge \( e' \in E(G) - E(P) \) incident with either \( x_{k-1} \) or \( x_k \). It is not difficult to see from the maximality of \( k \) that the subgraph of \( G \) formed by the edge set \( E(P) \cup \{ e, e' \} \) contains either \( C_n \) or \( F_n \) for some \( n \geq N \), contradicting the assumption.

\textbf{Remark.} For an ideal \( \mathcal{D} \) of digraphs (with respect to \( \subseteq_d \)), we do not know any necessary and sufficient condition to make \( (\mathcal{D}, \subseteq_d) \) a wqo.

\section{Counterexamples}

As we have seen from the previous section that \( (\mathcal{P}_n, \preceq) \) is a wqo. If \( \mathcal{Q}_n \) is the class of graphs \( G \) such that \( G \) has no induced subgraph \( P_n \), then a natural question is: is \( (\mathcal{Q}_n, \preceq) \) a wqo? It has been shown in [1] that the answer is yes if \( n \) is four. For \( n \) bigger than four, the following bad sequence (which is also mentioned in [1]) shows that the answer is no.

\textbf{Example 1.} For each \( n = 3, 4, \ldots \), let \( S_n \) be the graph on \( \{ x_1, \ldots, x_{2n} \} \) such that \( x_1 x_2, x_2 x_3, \ldots, x_{2n-1} x_{2n}, x_{2n} x_1 \) are edges of \( S_n \), and the other edges of \( S_n \) are all the pairs of the form \( x_2 x_2 \). It is not difficult to see that \( S_3, S_4, \ldots \) is a bad sequence. Moreover, for each \( n = 3, 4, \ldots \), \( S_n \) has no induced subgraphs \( 2K_2 \) (the disjoint union of two copies of \( K_2 \)) and \( \overline{2K_2} \) (the complement of \( 2K_2 \)).

It was conjectured in [1] that if \( P_5, \overline{P_5}, S_3, \) and \( \overline{S_3} \) are excluded, then we end up with a class of graphs well-quasi-ordered by \( \preceq \). However, our next example shows that even if \( 2K_2, \overline{2K_2}, C_5, \) and \( S_4 \) are also excluded (which is equivalent to excluding \( 2K_2, C_5, S_3, S_4, \ldots \) and their complements), there still is a bad sequence.
Example 2. Let $n \geq 2$ be an integer and let $V = \{a_1, \ldots, a_n, b_1, \ldots, b_{n+3}\}$. We first define two linear orderings on $V$

\[
b_1 \prec b_2 \prec b_3 \prec a_1 \prec b_4 \prec \cdots \prec a_i \\
\prec b_{i+3} \prec \cdots \prec a_{n-1} \prec b_{n+2} \prec b_{n+3} \prec a_n
\]

and

\[
b_{n+3} \prec b_{n+2} \prec b_{n+1} \prec a_n \prec b_n \prec \cdots \prec a_i \\
\prec b_i \prec \cdots \prec a_2 \prec b_2 \prec b_1 \prec a_1.
\]

We will write $x \preceq y$ ($x \preceq v$) if $x = y$ or $x \prec y$ ($x \prec v$). Let $V' = \{a_1', \ldots, a_n', b_1', b_{n+3}'\}$ and let $V'' = \{a_1'', \ldots, a_n'', b_1'', \ldots, b_{n+3}''\}$. We then define a graph $T_n$ on $V \cup V' \cup V''$ such that $E(T_n) = E \cup E' \cup E''$ where $E = \{uv: u \neq v \in V\}$, $E' = \{uv': u \in V, u' \in V'\}$, and $u \preceq v\}$ and $E'' = \{uv'': u \in V, u'' \in V'', u \preceq v\}$.

Claim 1. $T_2, T_3, \ldots$ is a bad sequence.

**Proof.** To prove this claim, we first observe that

1. $V'' \cup V''$ is a stable set and $V$ is a clique.
2. For each $x \in V(T_n)$, let $N(x)$ be the set of vertices of $T_n$ that are adjacent to $x$. Then for $v' \in V'$ and $v'' \in V''$, we have $N(v') = \{u \in V: u \preceq v\}$ and $N(v'') = \{u \in V: u \preceq v\}$. 
3. For every $x \in V' \cup V''$, $N(x)$ is a clique, but for every $x \in V$, $N(x)$ is not a clique.
4. For any $u, v \in V' \cup V''$, $N(u) = N(v)$ only if $u = v$ or $u = b_i'$, $v = b_{n+3}''$.
5. If $x_1 \prec x_2 \prec \cdots \prec x_k$ is a consecutive (with respect to $\prec$) set of vertices of $V$, then in the graph $T_n \setminus \{x_1, \ldots, x_{k-1}\}$, $N(x'_1) = \cdots = N(x'_k)$. The similar property is also valid for $\preceq$.

Now suppose there are indices $n < m$ such that $T_n \preceq T_m$. To avoid the notation problem, we refer the vertices of $T_m$ as $U = \{c_1, \ldots, c_m, d_1, \ldots, d_{m+3}\}$, $U' = \{u': u \in U\}$ and $U'' = \{u'': u \in V\}$. We also assume that $T_n$ is a real induced subgraph of $T_m$, not just isomorphic to an induced subgraph of $T_m$.

It follows from observation (3) that $V \subseteq U$. Next, we shall show that $V' \cup V''$ can be chosen properly so that it is a subset of $U' \cup U''$. For suppose there is a vertex $x \in (V' \cup V'') \cap U$, from observation (1) we know that $x$ is the only element contained in $(V' \cup V'') \cap U$. We shall prove that $x$ can be replaced by a vertex in $(U' \cup U'') - (V' \cup V'')$. Since $U$ is a clique, $x$ is adjacent to all vertices in $V$ and thus $x = b_i'$ or $b_{n+3}''$. Notice that $T_n$ has
only two vertices with $V$ as neighborhood, it follows that at least one of $d'_i$ and $d''_{m+3}$ is not included in $V'' \cup V'''$ (say $d'_i$) and hence $x$ can be replaced by $d'_i$ as we wanted.

Let $V'_0 = \{c'_i: c_i \in V\} \cup \{d'_i: d_i \in V\}$ and let $V''_0 \subseteq U''$ be defined similarly. We now show that $V'' \cup V'''$ can be replaced by $V'_0 \cup V''_0$. We shall say two vertices $x$ and $y$ of a graph $G$ are similar if $N(x) = N(y)$ in $G$. It is easy to check that this similar relation is an equivalence relation. Let $m' = |U|$ and $n' = |V'|$. Then from observation (4) we deduce that in the graph $T_m$, both $U'$ and $U''$ have $m'$ equivalence classes and $U' \cup U''$ has $2m' - 1$ equivalence classes. Let $W = U - V$. It is clear from observation (5) that in the graph $T_m \setminus W$, $U'$ and $U''$ have $m'$ equivalence classes, namely $X'_1, X'_1, X'_{n-1}, X''_{n-1}, X'_1 \cup X''_{n}$, where $X'_1$ and $X''_{n}$ are the equivalence classes containing $d'_i$ and $d''_{m+3}$ respectively. On the other hand, in the graph $T_m \setminus W$, $V'' \cup V'''$ has exactly $2n' - 1$ equivalence classes. Thus $U' \cup U''$ has exactly $2n' - 1$ equivalence classes in $T_m \setminus W$, namely $X'_1, X'_1, X'_{n-1}, X''_{n-1}, X'_1 \cup X''_{n}$. It follows that (i) the intersection of $V'' \cup V'''$ with each of these equivalence classes is a singleton except for $X'_1 \cup X''_{n}$, which is a set of size two; and (ii) for any $x, y \in U - W$, $x'$ and $y''$ must be contained in different of these equivalence classes unless $x = b_1$ and $y = b_{n+3}$. We also know from observation (5) that (iii) for any $x \neq y \in U - W$, $x'$ and $y''$, $x''$ and $y'$ are also respectively contained in different of these equivalence classes. From (ii), (iii), and the fact that $U' \cup U''$ has only $2n' - 1 = |V'_0| + |V''_0| - 1$ equivalence classes in $T_m \setminus W$ we deduce that (iv) the intersection of $V'_0 \cup V''_0$ with each of these equivalence classes is also a singleton except for $X'_1 \cup X''_{n}$, which is a set of size two. Thus we conclude from (i) and (iv) that $V'' \cup V'''$ can be replaced by $V'_0 \cup V''_0$ as we wanted.

Finally, we associate each graph $T_n$ with a quasi-ordering $Q_n = (V, \preceq)$ as follows: For distinct $x, y \in V$, we define $x \preceq y$ if and only if $N(x) \cap V \subseteq N(y) \cap V$. It is not difficult to see that $x \preceq y$ if and only if $x \preceq' y$ and $x \preceq'' y$. This ordering is illustrated by a diagram in Figure 1. Clearly, what we have shown actually is that $Q_n$ can be obtained from $Q_m$ by deleting $m - n$ elements. This is obviously impossible. Thus we conclude that $T_n$ is not an induced subgraph of $T_m$ and hence $T_2, T_3, \ldots$ is a bad sequence. 

**Claim 2.** For all integers $n \geq 2$,

(i) $V(T_n)$ can be partitioned into a clique and a stable set;
(ii) $T_n$ has no induced subgraph $2K_2, 2K_2$, or $C_5$;
(iii) the two bipartite subgraphs of $T_n$ formed by $E'$ and $E''$ have no induced subgraph $2K_2$;
(iv) the bipartite subgraph of $T_n$ formed by $E' \cup E''$ has no induced subgraph $C_6, 3K_2$, or $C_8$;
(v) $T_n$ has no induced subgraph $S_k$ or $\overline{S}_k$ for any $k \geq 3$. 


FIGURE 1. $Q_4$.

**Proof.** Claim 2(i) is clear because $(V, V' \cup V'')$ is such a partition. Thus (ii) follows since $2K_2$, $C_4 = 2K_2$ and $C_5$ do not have such a partition. Claim 2(iii) is trivial, and (iv) is a consequence of (iii). To prove (v) we observe that for each $S_k (k \geq 3)$, there is only one way to partition the vertices into a clique and a stable set. Thus (iv) implies that $T_n$ does not have induced subgraphs $S_3$, $S_3$, and $S_4 = S_4$. Since for all $k \geq 3$, $S_k$ has at least one of these three graphs as an induced subgraph, (v) follows. $\blacksquare$

**Remark.** Graphs satisfying (i) are called split graphs (see [3] for more information on these graphs). Example 2 shows that split graphs without induced $S_3$, $S_3$, and $S_4$ are not well-quasi-ordered by $\leq$.

Incidentally, a simple modification of Example 2 gives a negative answer to another question in [1]. Namely, there exists a bad sequence $T'_1, T'_2, \ldots$ such that no $T'_i$ has an induced subgraph $K_3$ or $P_8$.

**Example 3.** For every integer $n \geq 2$, let $T'_n$ be the subgraph of $T_n$ formed by $E' \cup E''$. Clearly $T'_n$ is a bipartite graph (not just $K_3$ free). Moreover, from Claim 2(iv) we know that $T'_n$ does not have an induced subgraph $3K_2$ (and hence $P_8$), $C_6$, or $C_8$.

**Claim 3.** $T'_2, T'_3, \ldots$ is a bad sequence.

**Proof.** We first observe that

1. $T'_n$ is a connected bipartite graph with the bipartition $(X, Y)$, where $X = V$ and $Y = V' \cup V''$.
2. $Y$ has a partition $(Y_1, Y_2)$ where $Y_1 = \{y_{11}, \ldots, y_{1k}\}$ and $Y_2 = \{y_{21}, \ldots, y_{2k}\}$ such that $N(y_{11}) \subseteq N(y_{12}) \subseteq \cdots \subseteq N(y_{1k})$ and $N(y_{21}) \subseteq N(y_{22}) \subseteq \cdots \subseteq N(y_{2k})$. But $X$ does not have such a partition.

Now if there exist indices $n < m$ such that $T'_i \leq T'_m$, by adopting the notation in the proof of Claim 1, we deduce from the above observations that $V \subseteq U$ and $V' \cup V'' \subseteq U' \cup U''$. The rest of the proof is the same as that of Claim 1. $\blacksquare$
Example 3 shows that it is not enough to exclude $K_3$ and $P_5$ to make a class of graphs to be well-quasi-ordered under $\preceq$. However, it is enough to exclude $K_3$ and $P_6$ as shown in [1]. We do not know whether it is enough to exclude $K_3$ and $P_7$ (or even $P_8$). Partial results will be given in the next section. Finally, we close this section by making the following conjecture.

**Conjecture.** Let $n \geq 5$ be an integer and let $\Sigma_n$ be the class of permutation graphs without induced $P_n$ or $\overline{P_n}$. Then $(\Sigma_n, \preceq)$ is a wqo.

We refer the reader to [3] for the definition and basic properties of permutation graphs. We would like to point out that an affirmative answer to this conjecture (even just for the case $n = 5$) would generalize the main result in [1], which says that the $P_4$-reducible graphs are well-quasi-ordered by $\preceq$.

### 4. BIPARTITE GRAPHS

In the last section of this paper, we are going to present two classes of bipartite graphs that are well-quasi-ordered by $\preceq$. Because of the close relationship between split graphs and bipartite graphs, our results can be converted very easily to propositions on split graphs. However, we shall not make the translations here because they are straightforward.

We assume that all the graphs mentioned in this section are bipartite graphs unless otherwise stated. If $G = (X, Y, E)$ is a bipartite graph, we define the bipartite complement (with respect to the bipartition $(X, Y)$) of $G$, denoted by $\overline{G}$, to be the bipartite graph $(X, Y, X \times Y - E)$. Let $J_1$ and $J_2$ be the graphs illustrated in Figure 2. It is not difficult to see that the bipartite complement of $J_1$ and $J_2$ is still $J_1$ and $J_2$, respectively. Let $\mathcal{H}$ be the class of bipartite graphs $G$ such that $G$ does not have an induced subgraph $P_3$, $J_1$, or $J_2$. We shall prove that

**Theorem 4.1.** $(\mathcal{H}, \preceq)$ is a wqo.

To prove Theorem 4.1, we first prove the following structure theorem on $\mathcal{H}$:

![Figure 2. $J_1$ and $J_2$.](image)
Theorem 4.2. Let $G$ be a connected bipartite graph with $|V(G)| > 1$. If the bipartite complement $\overline{G}$ of $G$ is also connected, then $G$ has an induced $P_7$, $J_1$, or $J_2$.

Proof. Without loss of generality, we assume that (1) both $G$ and $\overline{G}$ are connected; (2) $|V(G)| > 1$; and (3) for every vertex $v \in V(G)$, either $G \setminus v$ or $\overline{G} \setminus w$ is disconnected. We shall prove that if $G = (X, Y, E)$ has no induced $P_7$ and $J_1$, then $G$ has an induced $J_2$.

It is clear from (1) that $|X|, |Y| \geq 3$. Therefore from (3) we may assume, by taking the bipartite complement if necessary, that there are two vertices $y_1, y_2 \in Y$ such that both $G \setminus y_1$ and $G \setminus y_2$ are disconnected. Naturally, these two cut vertices $y_1$ and $y_2$ separate $G$ into three parts as follows: Let $T$ be a spanning tree of $G$ and let $P$ be the unique $y_1, y_2$ path in $T$. Let $e_i$ ($i = 1, 2$) be the edge in $P$ adjacent to $y_i$ and let $T_0, T_1,$ and $T_2$ be the three connected components of $T \setminus \{e_1, e_2\}$ such that $y_i \in V(T_i)$ ($i = 1, 2$). Let $(X_0, X_1, X_2)$ and $(Y_0, Y_1, Y_2)$ be the partitions of $X$ and $Y \setminus \{y_1, y_2\}$ respectively such that $X_i \cup Y_i = V(T_i) \setminus \{y_i\}$ ($i = 1, 2$). It is clear from the choice of $y_1, y_2$ that $X_1$ and $X_2$ are not empty. Let $G_0 = G \setminus (X_1 \cup X_2 \cup Y_1 \cup Y_2)$. It is obvious by looking at the spanning tree $T$ that $G_0$, $G_0 \setminus y_1$, and $G_0 \setminus y_2$ are connected.

Claim 1. All the induced paths between $y_1$ and $y_2$ are $P_3$; and in addition, there exists a vertex $x_0 \in X_0$ such that $x_0$ is adjacent to both $y_1$ and $y_2$.

For suppose there exists an induced path $P_k$ between $y_1$ and $y_2$ with $k \geq 5$. It is clear that this path is contained in $G_0$. Since $X_1$ and $X_2$ are not empty, it follows that this path can be extended to an induced path $P_{k+2}$ of $G$ that contains an induced $P_7$. The existence of $x_0$ follows from the fact that $G_0$ is connected.

Claim 2. For $i = 1, 2$, $y_i$ is adjacent to all the vertices in $X_i$.

For if there is a vertex (say) $x_1 \in X_1$ that is not adjacent to $y_1$, then it is not difficult to see that the shortest path from $x_1$ to $X_2$ contains an induced $P_7$.

Claim 3. $Y_1 = Y_2 = \emptyset$.

Suppose that $Y_1 \neq \emptyset$. Then $Y_2 = \emptyset$ for otherwise the shortest path between $Y_1$ and $Y_2$ contains an induced $P_7$. Consequently, $X_2 = \{x_2\}$ is a singleton because we know from (1), (2), and (3) that $G$ has no similar pairs. If there are vertices $y'_1 \in Y_1$ and $x'_1 \in X_1$ that are not adjacent, let $x_1 \in X_1$ such that $y'_1$ is adjacent to $x_1$. Then it is clear that $G$ induces a $J_1$ on $\{x_0, x_1, x'_1, x_2, y'_1, y_1, y_2\}$. Therefore $X_1 = \{x_1\}$ and $Y_1 = \{y_1\}$ are singletons since $G$ has no similar pairs. If there is a path $P_k$ between $y_1$ and $y_2$ with $k \geq 5$, it follows from Claim 1 that there exists a vertex $x \in V(P_k)$ such that $x$ is adjacent to both $y_1$ and $y_2$. Consequently, there is an induced $J_1$ of $G$ contained
in \(V(P) \cup \{x_1, x_2, y_1, y_2\}\). Thus every path (not just induced paths) from \(y_1\) to \(y_2\) is a \(P_3\). From the fact that \(G \setminus y_1, y_2\) are connected, we deduce that \(y_1\) and \(y_2\) have the same set \(X_0'\) of neighbors in \(G_0\). If \(Y_0 = \emptyset\), then \(X_0' = X_0\) (since \(G\) is connected) is a singleton (since \(G\) has no similar pairs), which implies that \(G = P_4\), contradicting (1) since \(P_4\) is disconnected. Therefore there exist vertices \(y_0 \in Y_0\) and \(x_0' \in X_0'\) that are adjacent. It follows that \(G\) has an induced \(J_1\) on \(\{x_0', x_1, x_2, y_0, y_1, y_1', y_2\}\).

Since \(G\) has no similar pairs, Claim 3 implies that \(X_1 = \{x_1\}\) and \(X_2 = \{x_2\}\) are singletons. Now \(G \setminus x_1\) and \(G \setminus x_2\) are connected, thus \(G_1 \setminus x_1\) and \(G_1 \setminus x_2\) are disconnected. Apply Claim 3 to \(G_1\) and \(x_1, x_2\), we conclude that \(y_1\) and \(y_2\) are adjacent to all the vertices in \(X_0\). Choose a vertex \(x_3 \in X_0\) of maximum degree. Since \(G\) is connected, there is a vertex \(y_3 \in Y_0\) not adjacent to \(x_3\). Let \(x_4\) be a vertex adjacent to \(y_3\). It is clear that \(x_4 \in X_0\). From the choice of \(x_3\), there exists a vertex \(y_4\) adjacent to \(x_3\) but not to \(x_4\). Therefore \(G\) has an induced \(J_2\) on \(\{x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\}\) as we wanted.

Let \(\emptyset\) be a class of graphs. A sequence \(G_1, G_2, \ldots\) of graphs in \(\emptyset\) is called a \textit{minimal bad sequence} if it is a bad sequence (with respect to \(\le\)) and there is no bad sequence \(G_1', G_2', \ldots\) of graphs in \(\emptyset\) with \(|G_1'| = |G_1|, \ldots, |G_{i-1}'| = |G_{i-1}|\) and \(|G_i| > |G_i'|\) for some \(i \geq 1\). It is clear that if \((\emptyset, \leq)\) has a bad sequence, then it has a minimal bad sequence.

**Proof of Theorem 4.1.** For if \((\mathcal{H}, \leq)\) is not a wqo, we take a minimal bad sequence \(G_1, G_2, \ldots\) from \(\mathcal{H}\). We say \(G\) is a component of \(G_i\) if \(G_i\) is disconnected and \(G\) is a connected component of \(G_i\), or if \(G_i\) is disconnected and \(G\) is a connected component of \(G_i\). Let \(\mathcal{H}'\) be the class of components of these graphs \(G_i\). Then we claim that \((\mathcal{H}', \leq)\) is not a wqo. For suppose it is. We define \(\mathcal{G}\) to be the set of pairs \((G, g)\), where \(G \in \mathcal{H}'\) and \(g \in \{0, 1\}\). We also define \((G, g) \leq (G', g')\) if \(G \leq G'\) and \(g = g'\). It is clear that \((\mathcal{G}, \leq)\) is a wqo since \((\mathcal{H}', \leq)\) is. For every positive integer \(i\), let \(G_{i_1}, \ldots, G_{i_k}\) be the components of \(G_i\). Let \(g_i = 1\) if \(G_i\) is connected and let \(g_i = 0\) if otherwise. From Lemma 2.4 we conclude that there are indices \(i < j\) such that \(((G_{i_1}, g_{i_1}), \ldots, (G_{i_k}, g_{i_k})) \leq (G_{j_1}, g_{j_1}), \ldots, (G_{j_k}, g_{j_k}))\), and hence \(G_i \leq G_j\), contradicting the assumption that \(G_1, G_2, \ldots\) is a bad sequence. Therefore, \((\mathcal{H}', \leq)\) has a bad sequence \(G_{i_1}, G_{i_2}, \ldots\). Choose \(i\) such that \(i = i_k = \min\{i_1, i_2, \ldots\}\). Then \(G_{i_1}, \ldots, G_{i_{i-1}}, G_{i_k}, G_{i_{i+1}}, \ldots\) is a bad sequence, contradicting the assumption that \(G_1, G_2, \ldots\) is a minimal bad sequence.

**Corollary 4.3 ([1]).** (i) Let \(\mathcal{G}_1\) be the class of graphs (not necessarily bipartite) without induced \(K_3\) and \(K_2 + 2K_1\) (the disjoint union of \(K_2\) and two copies of \(K_1\)). Then \((\mathcal{G}_1, \leq)\) is a wqo.

(ii) Let \(\mathcal{G}_2\) be the class of graphs (not necessarily bipartite) without induced \(K_3\) and \(P_5\). Then \((\mathcal{G}_2, \leq)\) is a wqo.

**Proof.** Let \(\mathcal{B}\) be the class of bipartite graphs. As observed in [1], \(G \in \mathcal{G}_1 - \mathcal{B}\) if and only if \(G = C_5\); and \(G \in \mathcal{G}_2 - \mathcal{B}\) if and only if every con-
nected component $G'$ of $G$ is a multiple $C_5$ (i.e., $V(G')$ can be partitioned into nonempty stable sets $V_1, \ldots, V_5$ such that the edges of $G'$ are precisely those $e = xy$ with $x \in V_i$ and $y \in V_{i+1}$ ($i = 1, \ldots, 5$) where $V_6 = V_1$). Since $P_2, J_1, \text{ and } J_2$ have induced subgraphs $P_3$ and $K_2 + 2K_1$, the results follow from Theorem 4.1.

In the following, we are going to present another class of bipartite graphs that are well-quasi-ordered under $\preceq$. Let $\mathcal{H}_n$ be the class of bipartite graphs without induced $P_n$ and its bipartite complement $\overline{P}_n$. A consequence of Example 3 is that $(\mathcal{H}_8, \preceq)$ is not a wqo. However, we will show that

**Theorem 4.4.** $(\mathcal{H}_6, \preceq)$ is a wqo.

We have no idea if $(\mathcal{H}_2, \preceq)$ is a wqo. We remark it here that $\overline{P}_2$ is again a $P_2$.

To prove Theorem 4.4, we shall need the next lemmas, which follow from Theorem 4.1 immediately.

**Lemma 4.5.** $(\mathcal{H}_4, \preceq)$ is a wqo.

**Lemma 4.6.** If $\mathcal{F}_0$ is the class of bipartite graphs without induced $2K_2$, then $(\mathcal{F}_0, \preceq)$ is a wqo.

For any two bipartite graphs $G_1 = (X_1, Y_1, E_1)$ and $G_2 = (X_2, Y_2, E_2)$, we define the join of these two graphs to be the bipartite graph $G_1 + G_2 = (X_1 \cup X_2, Y_1 \cup Y_2, E_0 \cup E_1 \cup E_2)$ where $E_0 = \{xy : x \in X_1, y \in Y_2\}$. Clearly, $G_1 + G_2 = G_2 + G_1$. Let $\mathcal{F}$ be the class of bipartite graphs $G$ such that either $G \in \mathcal{F}_0$, or $G$ or $\overline{G} \in \mathcal{H}_4$.

**Lemma 4.7.** $G \in \mathcal{H}_6$ if and only if $G$ can be constructed from graphs in $\mathcal{F}$ by a series of join operations.

**Proof.** The "if" part is clear, so we only need to show the "only if" part. Let $G \in \mathcal{H}_6 - \mathcal{F}$. We want to show that $G = G_1 + G_2$ for some graphs $G_1$ and $G_2$. Let $Z$ be the set of isolated vertices of $G$. If $Z \neq \emptyset$, it is clear that $G = G_1 + G_2$, where $G_1 = G \setminus Z$ and $G_2 = (Z, \emptyset, \emptyset)$. Thus we may assume that $Z = \emptyset$.

Let $G' = (X', Y', E')$ be an induced subgraph of $G$ such that its connected components $G'_i = (X'_i, Y'_i, E'_i)$ ($i = 1, \ldots, n$) are complete bipartite graphs. We choose this $G'$ with the property that (i) $n$ is maximized; and (ii) subject to (i), $|V(G')|$ is maximized. From the choice of $G$ we know that $n \geq 2$. Let $X_0 \subseteq X - X'$ be the set of vertices that are adjacent to at least one vertex in $Y'$ and let $Y_0$ be defined analogously. Let $A = X - (X' \cup X_0)$, $B = Y - (Y' \cup Y_0)$. It follows from the maximality of $n$ that there is no edge between $A$ and $B$. 

Claim 1. If \( x \in X_0 \) is adjacent to a vertex \( y \in Y' \), then \( x \) is adjacent to all the vertices in \( Y' \).

For if \( x \) is not adjacent to \( y' \in Y' \), let \( x' \in X' \). Since \( G_i' \) is a complete bipartite graph, \( x' \) is adjacent to both \( y \) and \( y' \). Let \( j \neq i \) (this \( j \) exists since \( n \geq 2 \)) and let \( x'' \in X_j', y'' \in Y_j' \). Then dependent on if \( x \) is adjacent to \( y'' \), \( G \) has an induced \( P_6 \) or \( \overline{P_6} \) on \( \{ x, x', x'', y, y', y'' \} \), contradicting the assumption.

Claim 2. For every \( x \in X_0 \), \( x \) is adjacent to all the vertices in \( Y' \).

It follows from Claim 1 and the maximality of \( V(G') \) that there exist \( i \neq j \) such that \( x \) is adjacent to all the vertices in \( Y'_i \cup Y'_j \). If \( x \) is not adjacent to some vertex \( y \in Y'_k \), then we may assume that \( y \) is contained in \( Y'_k \) for some \( k \neq i, j \). Take \( x_1 \in X'_i, x_k \in X'_j, y_i \in Y'_i \) and \( y_j \in Y'_j \). It is clear that \( G \) has an induced \( P_6 \) on \( \{ x, x_1, x_k, y, y_i, y_j \} \), contradicting the assumption again.

Similarly, we have

Claim 2'. For every \( y \in Y_0 \), \( y \) is adjacent to all the vertices in \( X' \).

Claim 3. If \( n \geq 3 \), then \( G \) is the join of two smaller graphs.

Proof of Theorem 4.4. It follows from Lemma 4.5 and Lemma 4.6 that we only need to show \((\mathcal{H}_6 - F, \leq)\) is a wqo. For suppose it is not. Take a minimal bad sequence \( G_1, G_2, \ldots \) from \((\mathcal{H}_6 - F, \leq)\). Because of Lemma 4.7, we may assume that each \( G_i \) is the join of \( G_{i1} \) and \( G_{i2} \). Let \( \mathcal{H} \) be the set of all these graphs \( G_i \). Then from Lemma 2.4 we deduce that \((\mathcal{H}, \leq)\) is not a wqo and thus there is a bad sequence \( G_{i_{j1}}, G_{i_{j2}}, \ldots \) contained in \( \mathcal{H} \). Choose \( k \geq 1 \) with \( i_k \) as small as possible. It is not difficult to see that \( G_1, \ldots, G_{i_{k-1}}, G_{i_k}, G_{i_{k+1}}, \ldots \) is a bad sequence, contradicting the assumption that \( G_1, G_2, \ldots \) is a minimal bad sequence.
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