Chordal Graphs, Interval Graphs, and wqo

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Abstract: Let $\preceq$ be the induced-minor relation. It is shown that, for every $t$, all chordal graphs of clique number at most $t$ are well-quasi-ordered by $\preceq$. On the other hand, if the bound on clique number is dropped, even the class of interval graphs is not well-quasi-ordered by $\preceq$. © 1998 John Wiley & Sons, Inc. J Graph Theory 28: 105–114, 1998

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1. INTRODUCTION

The graph-minors project of Robertson and Seymour motivates the question as to whether any class of graphs is well-quasi-ordered by any other graph-containment relation. This paper is concerned with a special graph-containment relation, the induced-minor relation. All graphs considered here will be simple, unless otherwise stated.

We start with the formal definition of an induced minor. Let $e = \{u, v\}$ be an edge of a graph $G$. The result of contracting $e$ is a new graph $G/e$, which is obtained from $G$ by deleting $u$ and $v$, and then adding a new vertex and making it adjacent to those vertices in $V(G) - \{u, v\}$ that are adjacent to at least one of $u$ and $v$. Clearly, this operation is the result of an ordinary contraction plus a simplification. We need to define contraction this way because we only consider simple graphs. It is
worth mentioning that our result in Section 3 also implies the corresponding result for nonsimple graphs because we allow labels. A graph $H$ is an induced minor of a graph $G$, written $H \preceq G$, if $H$ is isomorphic to a graph obtained from $G$ by deleting vertices and contracting edges.

Since an induced minor is a special minor, every class of graphs that is closed under minors is also closed under induced minors. But, in order to study the differences between minors and induced minors, it is natural to seek classes of graphs that are closed under induced minors yet not under minors. This leads to the concept of intersection graphs.

The intersection graph of sets $X_1, X_2, \ldots, X_n$ is the graph $G$ with vertex set $\{X_1, X_2, \ldots, X_n\}$ for which $\{X_i, X_j\}$ is an edge if and only if $i \neq j$ and $X_i \cap X_j \neq \emptyset$. Observe that, if $e = \{X_1, X_2\}$ is an edge, then $G/e$ is precisely the intersection graph of $X_1 \cup X_2, X_3, \ldots, X_n$. Let $\Sigma$ be a family of sets and let $G(\Sigma)$ be the family of intersection graphs of members of $\Sigma$. Clearly, if $X \cup Y \in \Sigma$ is always implied by $X \in \Sigma, Y \in \Sigma$, and $X \cap Y \neq \emptyset$, then $G(\Sigma)$ is closed under induced minors.

In this paper, we consider two classes of intersection graphs. Let $T$ be an infinite tree that contains all finite trees as subgraphs. Such a tree obviously exists. Let $\Sigma$ be the set of vertex sets of finite subtrees of $T$. Then $G(\Sigma)$ is exactly the class of chordal graphs, that is, graphs for which every circuit of size at least four has a chord. We will also consider a class of special chordal graphs called interval graphs. These are the intersection graphs of vertex sets of subpaths of an infinite path. They are called interval graphs because they are exactly the intersection graphs of intervals of the real line. From the discussion of the previous paragraph, it is clear that both the class of chordal graphs and the class of interval graphs are closed under $\preceq$. The reader is referred to [2] for basic properties of chordal graphs and interval graphs.

A binary relation $\leq$ on a set $Q$ is a quasi-order if $\leq$ is reflexive and transitive. It is a well-quasi-order (or a wqo) if every infinite sequence $q_1, q_2, \ldots$ of members of $Q$ has indices $i$ and $j$ such that $i < j$ and $q_i \leq q_j$. Clearly, $\preceq$ is a quasi-order on the class of all graphs. However, it is not a wqo, as shown by the sequence of graphs $\bar{C}_3, \bar{C}_4, \ldots$, where $\bar{C}_n$ is the complement of the circuit on $n$ vertices.

In the next section, as the first result of the paper, we will present an infinite sequence of graphs which shows that interval graphs are not well quasi-ordered by $\preceq$. The second result of this paper involves cliques, that is, complete subgraphs. The size of a clique is its number of vertices. We prove in Section 3 that, for each positive integer $t$, the class of chordal graphs that do not have cliques of size $t$ is well quasi-ordered by $\preceq$. In fact, what we are going to prove is much stronger. We will allow labels and orientations on all cliques.

We conclude this section by mentioning two related results. In [5], it is proved that the class of series-parallel graphs is well quasi-ordered by $\preceq$, yet the class of planar graphs is not. The other result is one in [1] which says that a graph is an interval graph if and only if it does not contain any one of five graphs as an induced minor.
2. INTERVAL GRAPHS

The goal of this section is to show that the class of interval graphs is not well-quasi-ordered by \(\preceq\). Clearly, it is enough to construct an infinite antichain, that is, an infinite set of interval graphs none of which is an induced minor of another.

**The construction.** Let \(n\) be an integer exceeding two. Let \(S_n\) be the set of closed intervals
\[
\{[i, i] \mid i = \pm1, \pm2, \ldots, \pm2n\}.
\]
Let \(T_n\) be the set of closed intervals
\[
\begin{align*}
&\{[-2, 2], [-4, 1], [-2n + 3, 2n], [-2n + 1, 2n - 1]\}; \\
&\{[-2i + 1, 2i + 1] \mid i = 1, 2, \ldots, n - 2\}; \text{ and} \\
&\{[-2i, 2i - 2] \mid i = 3, 4, \ldots, n\}.
\end{align*}
\]
The following diagram may help the reader to visualize intervals in \(T_6\), where, in order to illustrate these intervals, the real line is folded up.

![Diagram of intervals in T6](image)

Let \(G_n\) be the intersection graph of intervals in \(S_n \cup T_n\).

**Theorem 1.** The set of graphs \(G_n\) is an antichain with respect to \(\preceq\).

The rest of this section is to prove Theorem 1. We start with some propositions. The first of these is obvious.

**Proposition 1.** If \(e = \{x, y\}\) is an edge of a graph \(G\) such that every vertex adjacent to \(x\) is either \(y\) or adjacent to \(y\), then \(G/e\) is isomorphic to \(G - x\).

This proposition suggests the following definition. An edge \(\{x, y\}\) of a graph \(G\) is **contractible** if there are vertices \(x'\) and \(y'\) in \(V(G) - \{x, y\}\) such that \(\{x, x'\}\) and \(\{y, y'\}\) are edges but \(\{x, y'\}\) and \(\{x', y\}\) are not. The next proposition is immediate from Proposition 1.

**Proposition 2.** If \(H \preceq G\), then \(H\) can be obtained from \(G\) by deleting vertices and contracting contractible edges.

Let \(e = \{x, y\}\) be a non-contractible edge of a graph \(G\). If \(z\) is a vertex other than \(x\) and \(y\), it is easy to see that \(e\) is still not contractible in \(G - z\). Similarly, if \(f\) is an edge not incident with \(e\), then \(e\) is not contractible in \(G/f\) either. Finally,
if \( f = \{x, z\} \) is an edge other than \( e \), and \( u \) is the new vertex of \( G/f \), then the
new edge \( \{u, y\} \) is not contractible in \( G/f \) unless \( \{y, z\} \) is a contractible edge
of \( G \). To sum these up, we conclude that every contractible edge of an induced
minor of \( G \) naturally corresponds to a contractible edge of \( G \). In fact, the same
reasoning proves the following stronger statement. For any graph \( G \), let \( G^* \) denote
the subgraph of \( G \) induced by its contractible edges.

**Proposition 3.** If \( H = (G - X)/F \) for a set \( X \) of vertices of \( G \) and a set \( F \) of
contractible edges of \( G \), then \( H^* \) is a subgraph of \( (G^* - X)/F \).

We say two intervals cross if they meet but neither is a subset of the other. Let
\( H_n \) be the graph with vertex set \( T_n \) such that \( \{x, y\} \) is an edge if and only if \( x \) and
\( y \) cross. From the definition of \( T_n \), it is not difficult to verify the following.

**Proposition 4.** \( H_n \) consists of two vertex-disjoint triangles and a path
joining them.

![FIGURE 2. The graph \( H_6 \).](image)

Since every two distinct members of \( T_n \) are adjacent in \( G_n \), we may consider
\( H_n \) as a subgraph of \( G_n \). Notice that every contractible edge of an interval graph
must be incident with two crossing intervals. Thus every contractible edge of \( G_n \) is
an edge of \( H_n \). Conversely, for every edge \( \{x, y\} \) of \( H_n \), by choosing appropriate
\( x' \) and \( y' \) from \( S_n \), it is easy to see that \( \{x, y\} \) is contractible in \( G_n \). Therefore, we
conclude the following.

**Proposition 5.** \( G_n^* = H_n \).

Recall that, if \( e = \{x, y\} \) is an edge of an interval graph \( G \), then \( G/e \) is the
intersection graph of intervals in \( (V(G) - \{x, y\}) \cup \{x \cup y\} \). This fact, together
with the definition of \( T_n \), implies the following.

**Proposition 6.** If \( e \) is a contractible edge of \( G_n \), then every connected component
of \( (G_n/e)^* \) has at most one triangle.

We are now ready to prove Theorem 1.

**Proof of Theorem 1.** Suppose \( G_m \) is an induced minor of \( G_n \) for some \( m < n \).
Then, by Propositions 2 and 5, \( G_m \) can be obtained from \( G_n \) by deleting a set of
vertices of \( G_n \) and contracting a set \( F \) of edges of \( H_n \). Clearly, by Proposition 4,
\( H_m \) is not a subgraph of \( H_n \). Thus we deduce from Proposition 3 that there exists
an edge \( e \) in \( F \), and \( H_m \) is a subgraph of \( (G_n/e)^* \). However, \( H_m \) is connected with
two triangles. But, by Proposition 6, every connected component of \( (G_n/e)^* \) has
at most one triangle. This contradiction proves that no \( G_m \) is an induced minor of
another \( G_n \) and thereby finishes the proof of Theorem 1.
Remark. A graph $G$ is called a split graph if it has an independent set $S$ of vertices such that $G - S$ is a clique. It is well known [2] that $G$ is a split graph if and only if both $G$ and its complement are chordal. From the definitions of $S_n$ and $T_n$, we deduce easily that $S_n$ is independent and $G_n - S_n$ is a clique. Hence each $G_n$ is a split graph. Therefore, Theorem 1 actually implies that the class of split interval graphs is not well-quasi-ordered by $\preceq$.

3. CHORDAL GRAPHS

The goal of this section is to prove that chordal graphs of bounded clique size are well-quasi-ordered by $\preceq$. In order to prove this statement, we have to prove something stronger so that our reduction will work.

A graph is fully oriented if the vertex set of every clique is linearly ordered. A fully oriented graph can be considered as a generalization of an ordinary oriented graph for which only the vertex set of every clique of size two is linearly ordered. Notice that we do not have any restriction on the relationship between the linear orders on different cliques.

Let $Q$ be a set. A $Q$-labeling of a graph $G$ is a mapping $f$ from the set of all cliques of $G$ to $Q$. The pair $(G, f)$ is called a $Q$-labeled graph. The objects we are going to consider, instead of ordinary graphs, will be $Q$-labeled fully oriented graphs. Using these new objects will produce a stronger result. But the main reason for introducing them is to make the reduction in our proof work.

Next, we extend the operations of deleting vertices and contracting edges on ordinary graphs to operations on $Q$-labeled, fully oriented, chordal graphs. Let $G$ be such a graph. If $v$ is a vertex, then deleting $v$ from $G$ is defined in the natural way. Cliques containing $v$, together with their orientations and labels, will disappear. All cliques of $G - v$ will inherit their orientations and labels from $G$. Now let $e = \{x, y\}$ be an edge of $G$ and let $z$ be the new vertex of $G/e$. Cliques not containing $z$ will inherit their orientations and labels from $G$. For a clique $K$ containing $z$, it is not difficult to see that at least one of $K - z + x$ and $K - z + y$ is a clique of $G$ since $G$ is a chordal graph. In this case, we choose one of $K - z + x$ and $K - z + y$ that is a clique (if both are, we can choose either one) and give $K$ the orientation and label of that clique.

The next figure illustrates the idea of this modified contraction. When the edge $\{x, y\}$ is contracted, the cliques not containing the new vertex $z$, which are $u, v, and uv$, inherit their orientations and labels from $G$. For cliques $zu$ and $zuv$, there are no other choices, they must inherit their orientations and labels from $xu$ and $xuv$, respectively. For cliques $z$ and $zv$, there are choices: $z$ can inherit the label of $x$ or the label of $y$ (they have the same trivial orientation). Similarly, $zv$ can inherit the orientation and label of $xv$, or the orientation and label of $yv$. Notice that although the orientation and labeling of the clique $zuv$ are determined, as a subgraph of it, the clique $zv$ still has the freedom to choose its own labeling and orientation.
Let \( \leq \) be a binary relation on \( Q \) and let \((H, h)\) and \((G, g)\) be \( Q \)-labeled fully oriented chordal graphs. We define \((H, h) \preceq (G, g)\) if some \((H, g')\) can be obtained from \((G, g)\) by deleting vertices and contracting edges such that \(h(K) \leq g'(K)\) for all cliques \( K \) of \( H \). It is not difficult to verify that the following is an equivalent definition. \((H, h) \preceq (G, g)\) if there are disjoint subsets \(X_v\) of \(V(G)\) for all vertices \( v \) of \( H \) such that

1. the subgraph of \( G \) induced by each \( X_v \) is connected;
2. \( H \) has an edge between two distinct vertices \( u \) and \( v \) if and only if \( G \) has at least one edge between \( X_u \) and \( X_v \);
3. for every clique \( K \) of \( H \), with vertices ordered as \( v_1, v_2, \ldots, v_k \), there is a clique \( K' \) of \( G \) with vertices ordered as \( x_1, x_2, \ldots, x_k \) such that \( x_i \in X_{v_i} \) for all \( i \) and \( h(K) \leq g(K') \).

We are now ready to state the main result of this section.

**Theorem 2.** Let \((Q, \leq)\) be a wqo and let \( t \) be a positive integer. Then the class of all \( Q \)-labeled fully oriented chordal graphs without cliques of size \( t + 1 \) is well-quasi-ordered by \( \leq \).

If we ignore the orientations and labels, Theorem 2 is exactly what we stated in the abstract of this paper: chordal graphs of bounded clique size are well-quasi-ordered by the induced-minor relation.

Before proving Theorem 2, we need some preparation. Let \((Q, \leq)\) be a quasi order. Let \( Q^* \) be the set of finite sequences of members of \( Q \). Then \( \leq \) can be extended to a quasi-order \( \leq^* \) on \( Q^* \) as follows. For any two members \( p = \langle p_1, p_2, \ldots, p_m \rangle \) and \( q = \langle q_1, q_2, \ldots, q_n \rangle \) of \( Q^* \), let \( p \leq^* q \) if there are indices \( i_1, i_2, \ldots, i_m \) with \( 1 \leq i_1 < i_2 < \cdots < i_m \leq n \) such that \( p_j \leq q_{i_j} \) for all \( j \). The following is a well-known theorem of Higman [3].

**Higman’s Theorem.** If \((Q, \leq)\) is a wqo, then so is \((Q^*, \leq^*)\).

The following is an obvious fact about wqo that we will use.

**Proposition 7.** Let \((Q, \leq)\) be a quasi-order and let \( Q_i \subseteq Q \) for \( i = 1, 2, \ldots, n \). If each \((Q_i, \leq)\) is a wqo, then \((\bigcup Q_i, \leq)\) is also a wqo.

The next proposition follows immediately from the last two results.

**Proposition 8.** Suppose \((Q_i, \leq_i)\) is a wqo for all \( i = 1, 2, \ldots, n \). Then \((Q_1 \times Q_2 \times \cdots \times Q_n, \leq)\) is also a wqo, where \((p_1, p_2, \ldots, p_n) \leq (q_1, q_2, \ldots, q_n)\) means \( p_i \leq_i q_i \) for all \( i \).
A cut of a connected graph $G$ is a set $X$ of vertices such that $G - X$ is disconnected. A cut is minimal if none of its proper subsets is a cut. If $G$ is a connected chordal graph, it is well known, and also easy to verify, that every minimal cut induces a clique. Suppose $R$ is a clique of a chordal graph $G$ and $X$ is a minimal cut with $V(R) \nsubseteq X$. Then we denote by $G(X, R)$ the subgraph of $G$ induced by $V(G') \cup X$, where $G'$ is the connected component of $G - X$ that contains $R - X$.

In this paper, we call a graph $s$-connected if it has at least $s$ vertices and has no cuts of size less than $s$. This is almost the usual definition except we consider a clique of size $s$ to be $s$-connected.

For a positive integer $s$, let $R$ be a clique of size $s$ in an $s$-connected chordal graph $G$. Then a cut $X$ of size $s$ is tight for $R$ if either $V(R) = X$, or $V(R)$ is not a cut and $G(X, R)$ has no cuts of size $s$ that separate $R$ from $X$. The last condition means that, for every cut $Y$ of $G(X, R)$ of size $s$, both $R - Y$ and $X - Y$ are contained in the same component of $G(X, R) - Y$. Let $\mathcal{C}(R)$ denote the set of cuts of $G$ that are tight for $R$. Clearly, $\mathcal{C}(R)$ is not empty, unless $G$ has no cuts of size $s$ when $G = R$ or $G$ is $(s + 1)$-connected. In fact, it is not difficult to verify the following.

**Proposition 9.** Suppose $R$ is a clique of size $s$ in an $s$-connected chordal graph $G$ with $G \neq R$. If $V(R)$ is not a cut, then the intersection of $G(X, R)$ over all $X$ in $\mathcal{C}(R)$ is $(s + 1)$-connected.

The basic idea of our proof of Theorem 2 is the ‘‘minimal bad sequence’’ argument of Nash-Williams [4]. Let $(Q, \leq)$ be a quasi-order without infinite descending chains. An infinite sequence $q_1, q_2, \cdots$ of members of $Q$ is a minimal bad sequence if no two distinct $q_i$’s are comparable, and the set $\{q : q < q_i \text{ for some } i\}$ is well-quasi-ordered by $\preceq$. The following can be found in [4] and [5].

**Proposition 10.** If a quasi-order without infinite descending chains is not a wqo, then it has a minimal bad sequence.

The following is an obvious property of minimal bad sequences.

**Proposition 11.** Every infinite subsequence of a minimal bad sequence is also a minimal bad sequence.

In the proof of Theorem 2, we will consider rooted ($Q$-labeled, fully oriented, chordal) graphs, that is, graphs with a specified clique called the root. One way to think of rooted graphs is to think of having a special label on the clique that is the root. Then the induced-minor relation can be extended to rooted graphs as follows. To ensure that the induced minor will also be a rooted graph, vertices in the root clique cannot be deleted, and, when choosing a clique during contraction, this root clique should always be chosen. Obviously, Theorem 2 follows if we can prove that rooted graphs are well-quasi-ordered. On the other hand, Theorem 2 implies its rooted version, which makes the theorem and its rooted version equivalent. To see the equivalence, consider $Q' = Q \cup \{r\}$, where $r$ is an element not in $Q$. Extend $\leq$ to $\leq'$ on $Q'$ in such a way that $r$ is not comparable with any element in $Q$. Then
it is clear that \((Q', \leq')\) is a wqo provided \((Q, \leq)\) is. Suppose we consider those \(Q'\)-labeled graphs for which there is exactly one clique having label \(r\). Then we deduce the rooted version of Theorem 2 from Theorem 2 itself immediately.

**Proof of Theorem 2.** Let \(s\) be in \(\{0, 1, \ldots, t\}\). Let \(G_s\) be the class of \(s\)-connected, fully oriented, chordal graphs that do not have cliques of size \(t + 1\). Also, let \(G_s(Q)\) be the class of all \(Q\)-labeled members of \(G_s\). We need to show that \(G_0(Q)\) is well-quasi-ordered by \(\preceq\).

Let \(G^r_s\) be the class of rooted members of \(G_s\) for which the root is a clique of size \(s\). Notice that \(G^r_0\) actually is \(G_0\). As before, let \(G^r_s(Q)\) be the class of all \(Q\)-labeled members of \(G^r_s\).

Observe that the only \(t\)-connected chordal graph with no cliques of size \(t + 1\) is a clique of size \(t\). It follows that \(G^r_t\) has only finitely many members. Thus we deduce from Propositions 7 and 8 that \((G^r_t(Q), \preceq)\) is a wqo for every wqo \((Q, \preceq)\). In the following, we will prove that

if \(s > 0\) and \((G^r_s(Q), \preceq)\) is a wqo for every wqo \((Q, \preceq)\),

then \((G^r_{s-1}(Q), \preceq)\) is also a wqo for every wqo \((Q, \preceq)\). (1)

It is clear that (1) completes the proof of Theorem 2.

If \(s = 1\), then (1) follows easily from Higman’s Theorem. From now on, we assume that \(s > 1\) and thus \(t > 1\).

Suppose \((G^r_{s-1}(Q), \preceq)\) is not a wqo. Then, by Proposition 10, there is a minimal bad sequence \(\{(G_i, g_i) : i \geq 1\}\) of members of \(G^r_{s-1}(Q)\). For each \(i\), let \(R_i\) be the root of \(G_i\). We consider two cases.

Suppose there are infinitely many \(i\)’s for which \(V(R_i)\) is a cut of \(G_i\). By Proposition 11, we may assume for all \(i\) that \(V(R_i)\) is a cut of \(G_i\). Let \(G^i_{1,1}\) be \(G_i - V(G^r_i)\) and \(G^i_{1,2}\) be the subgraph of \(G_i\) induced by \(V(R_i) \cup V(G^r_i)\). Then \(G^i_{1,1}\) and \(G^i_{1,2}\) have the same root \(R_i\) and they are both proper induced minors of \(G_i\). For \(j = 1, 2\), let \(g^i_{1,j}\) be the restriction of \(g_i\) to \(G^i_{1,j}\). It is clear that \((G^i_{1,j}, g^i_{1,j}) \preceq (G^i_{1}, g^i_{1})\) for all \(i\) and \(j\). By the choice of \(\{(G_i, g_i) : i \geq 1\}\), the class of all \((G^i_{1,j}, g^i_{1,j})\) is well-quasi-ordered by \(\preceq\). Then, by Proposition 8, for some \(i < i'\) we have \((G^i_{1,1}, g^i_{1,1}) \preceq (G^i'_{1,1}, g^i'_{1,1})\) and \((G^i_{1,2}, g^i_{1,2}) \preceq (G^i'_{1,2}, g^i'_{1,2})\), which clearly implies \((G^i_{1}, g^i_{1}) \preceq (G^i', g^i')\). This is a contradiction since \(\{(G_i, g_i) : i \geq 1\}\) is a bad sequence.

Next, we consider the case when there are only finitely many \(i\)’s for which \(V(R_i)\) is a cut of \(G_i\). By Proposition 11, we may assume that no \(V(R_i)\) is a cut of \(G_i\). In addition, from Propositions 11 and 8, we may assume that each \(G_i\) has at least \(s\) vertices. Then we deduce from Proposition 9 that \(H_i\), the intersection of \(G_i(X, R_i)\) over all \(X\) in \(C(R_i)\), is an \(s\)-connected graph. Clearly, we can think of \(H_i\) as being obtained from \(G_i\) by deleting vertices. Therefore, \(H_i\) actually is a fully oriented graph and thus it is a member of \(G_s\). Moreover, since \(H_i\) is an \(s\)-connected chordal graph, it must have cliques of size \(s\). Take any such clique and declare that to be the root of \(H_i\). This declaration makes \(H_i\) a member of \(G^r_s\). Notice that the new root has nothing to do with \(R_i\), the root of \(G_i\). We do not really need the new root in our
reasoning, the only reason we define it is to make our statements more consistent with the assumption of (1).

Now we consider the rest of $G_i$. For each $X$ in $C(R_i)$, if $G'_i$ is the connected component of $G_i - X$ that contains $R_i - X$, then let $G_{i,X}$ be $G_i - V(G'_i)$. It is not difficult to see that $G_i$ is the union of $H_i$ and $G_{i,X}$ over all $X$ in $C(R_i)$. Moreover, the intersection of $H_i$ and each $G_{i,X}$ is the clique with vertex set $X$, which is denoted by $K_X$. Let us declare the root of $G_{i,X}$ to be $K_X$ and let the restriction of $g_i$ to $G_{i,X}$ be $g_{i,X}$. In order to use the assumption of (1), we need to show that the class $\mathcal{L}$ of all labeled graphs $(G_{i,X}, g_{i,X})$ are well-quasi-ordered by $\preceq$. This does not follow from Proposition 10 immediately because, according to our definition, $(G_{i,X}, g_{i,X})$ is not a rooted induced minor of $(G_i, g_i)$. The problem is that the root $R_i$ of $G_i$ is not inherited by $G_{i,X}$.

This problem can be fixed as follows. Since $H_i$ is $s$-connected, there are $s - 1$ vertex-disjoint paths linking $R_i$ and $K_X$. By deleting vertices not in $G_{i,X}$ or these paths, and then contracting edges in these paths, we get a rooted induced minor $(G'_{i,X}, g'_{i,X})$. It is easily seen that the only possible difference between $(G_{i,X}, g_{i,X})$ and $(G'_{i,X}, g'_{i,X})$ is that their roots may have different orientations and different labels. This close connection between these graphs, together with Proposition 10, will enable us to deduce that $(\mathcal{L}, \preceq)$ is a wqo.

Let $G$ be a member of $G'_{s-1}$ with root $R$. If the orientation of $R$, that is, the linear order of vertices of $R$, is $x_1, x_2, \ldots, x_{s-1}$, then, for any permutation $\pi$ of \{1, 2, $\ldots$, $s - 1$\}, let $\pi(G)$ denote the graph obtained from $G$ by changing the orientation of its root to $x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(s-1)}$. Let us consider $\mathcal{L}'$, the class of all labeled graphs $(G'_{i,X}, g'_{i,X})$. By Proposition 10, $\mathcal{L}'$ is well-quasi-ordered by $\preceq$. Consequently, for any given $\pi$, the class of all labeled graphs $(\pi(G'_{i,X}), g'_{i,X})$, which is denoted by $\pi(\mathcal{L}')$, is also well-quasi-ordered by $\preceq$. Therefore, by Proposition 7, the union of all $\pi(\mathcal{L}')$ is well-quasi-ordered by $\preceq$. In particular, the class of all labeled graphs $(G_{i,X}, g_{i,X})$ is well-quasi-ordered by $\preceq$. Now, by considering all pairs $(g_i(K_X), (G_{i,X}, g_{i,X}))$, we deduce from Proposition 8 that $(\mathcal{L}, \preceq)$ is a wqo.

Recall that $H_i$ is a member of $G'_s$. Now we encode all information about $G_i$ in $H_i$ by defining a new labeling $h_i$ of $H_i$. First, let $r$ be a new element and let $h_i(R_i) = (r, g_i(R_i))$. For cliques $K_X$ with $X$ in $C(R_i)$, let $h_i(K_X) = (G_{i,X}, g_{i,X})$. Finally, for all other cliques $K$, let $h_i(K) = g_i(K)$. The range $(\{r\} \times Q) \cup \mathcal{L} \cup Q$ of $h_i$ is denoted by $Q'$ and an ordering $\preceq'$ on $Q'$ is defined as follows: two members of $Q'$ are comparable if and only if they are of the same type and are comparable with respect to the ordering of that type. By Propositions 7 and 8, $(Q', \preceq')$ is a wqo. It follows from the assumption of (1) that, for some $i < i'$, we have $(H_i, h_i) \preceq (H_{i'}, h_{i'})$. But then, it is routine to verify that $(G_i, g_i) \preceq (G_{i'}, g_{i'})$. This is a contradiction, since $\{(G_i, g_i) : i \geq 1\}$ is a bad sequence, and this contradiction finishes the proof of Theorem 2.

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