Open problems of GRASTA 2017: the 6th Workshop on GRAph Searching, Theory and Applications

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1 Steve Alpern: A Simple Open Problem in Search Games on an Interval

The Princess and Monster Game on a network is a zero-sum game played between a Searcher (Monster, or Cop) and a mobile Hider (Princess, or Robber). The network is treated as a metric space, where each edge has a given length and distance is the length of the shortest path. We assume both players move with unit speed and no other restrictions, so that they can for instance turn around in the middle of an edge without slowing down. If the Searcher chooses path $S(t), t \geq 0$, and the Hider choose a path $H(t), t \geq 0$ (including their starting points $S(0)$ and $H(0)$), the payoff to the maximizing Hider is the capture time $T = T(S, H) = \min\{t : S(t) = H(t)\}$. A mixed strategy is simply a probability distribution over these pure strategies. Now suppose that the network is simply the unit interval $[0, 1]$. What is the solution to this game? That is, what are the optimal mixed strategies for each player, and what is the Value V of the game (the expected value of T with best play on both sides)?

An example of a strategy for the Searcher would be an equiprobable mixture of left to right, $S_+(t) = t$, and right to left, $S_-(t) = 1 - t$. The optimal response to this is for the Hider to start at $1/2$ and stay there until just before the Searcher gets there, say till time $1/2 - \varepsilon$. At this time the Hider equiprobably goes to 0 or to 1. If the Hider is lucky, the capture time is $T = 1$; if unlucky, the capture time is $T = 1/2 - \varepsilon/2$. Thus the expected value of T is $3/4 - \varepsilon/4$. But the Searcher has better strategies, so $V < 3/4$.

For references, see [1, 2].

References

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2 Anthony Bonato: Some topological questions and conjectures on Cops and Robbers

It is known (see [1]) that planar graphs (graphs of genus 0) have cop number at most 3. Nancy Clark, in [2], proved that outerplanar graphs have cop number at most 2.

Question: Can we characterize planar (outer-planar) graphs with cop number 1,2, and 3? Is the dodecahedron the unique smallest order planar 3-cop-win graph?

Concerning graphs of higher genus, Schroeder's conjecture states that if G has genus k, then $c(G) \leq k+3$. In regards to this conjecture, the following is known: The conjecture is true for $k = 0$; In [3] it was shown that it is true for $k = 1$ (toroidal graphs); [4] showed that $c(G) \leq 2k + 3$; and in [3] it was shown that $c(G) \leq \lfloor \frac{3k}{2} \rfloor + 3$.

Concerning the capture time of planar graphs, in [6], it was shown that if G is a connected planar graph and $k \geq 12\sqrt{n}$ than $\text{capt}_k(G) \leq 6 \cdot \text{rad}(G) \log n$. The proof uses the Planar Separator Theorem [5], works also if robber has infinite speed, and generalizes to higher genus (only k changes, not the bound). Moreover, in [6], it was shown that if G is a connected planar graph of order n, then $\text{capt}_3(G) \leq (\text{diam}(G) + 1)n$.

Question: Can we give bounds on $\text{capt}_2(G)$ if G is outerplanar? Are there examples of planar graphs with large 2,3, or even $\sqrt{n}$-capture times? Can we give an example of a planar G with $\text{capt}_{2,3}(G) = \Theta(n^2)$?.

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3 Joshua Erde: The Angel and Devil game

Statement of Problem The Angel and Devil game is a two player game played on a rooted infinite directed graph G. In his turn the first player, the *Devil*, chooses a vertex to block, which remains blocked for the rest of the game. The second player, the *Angel*, starts at the root r and in his turn can move to any unblocked square that is at (graph-)distance at most p away from his current position, we call p the *power* of the Angel. The players alternate turns, and the Devil wins if he can trap the Angel, that is if he can force the Angel into a position where he has no available moves. The Angel wins if he has a strategy to keep moving forever.

Originally this game was considered in the undirected case on the l_∞ grid $\mathbb{Z}^d$, which is the strong product of d double rays. A number of authors [3, 6, 7, 8] independently showed that the angel of power 2 wins in $\mathbb{Z}^2$. Blass and Conway [1] showed that if the Angel must increase his y coordinate in each move then the Devil can beat the Angle of **any** power in $\mathbb{Z}^2$. Bollobás and Leader [2] considered an Angel in $\mathbb{Z}^3$ that must always increase his z coordinate. They noted that this was equivalent to playing the game in $\mathbb{Z}^2$ with a weaker devil, who on his turn cannot block a vertex forever, but only at some specific time step in the future. They called this game the *Time-Bomb game* and conjectured

Conjecture: [Bollobás and Leader] The Angel of power one wins the Time-Bomb game in $\mathbb{Z}^2$

Recently the Angel-Devil game has been considered on the directed graphs $\mathcal{P}^d$ obtained by taking the cartesian product of d directed rays. The results of [1] shows that the Devil wins against an Angel of any power in $\mathcal{P}^2$, and Clarke, Finbow, Fitzpatrick, Messinger and Nowakowski [4] showed, by considering an equivalent game, that the Devil wins against the Angel of power 1 in $\mathcal{P}^3$. However, Cranston, Kinnersley, Milans, Puleo and West [5] showed that for $k \geq 14$ the Angel of power one wins on $\mathcal{P}^k$.

Question: What is the smallest k such that the Angel of power one wins on $\mathcal{P}^k$?

Similar methods as in [5] would show that a positive answer to the above Conjecture would imply $k \leq 8$.

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4 Fedor V. Fomin: Complexity of connected search when the number of searchers is small

In the connected graph searching problem the task is to identify the minimum number of searchers sufficient to clear the graph such that at every step of searching the cleared area is connected. (For a formal definition, please see [1].) While (not connected) graph searching problem is fixed-parameter tractable with a standard parameterization by the number of searchers, for the connected search problem the situation is much more obscure. Can it be that the problem is NP-hard already for small values of searchers like 3 or 4? Similar questions are open for the monotone version of connected search.

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5 Thanasis Kehagias: Multiplayer pursuit on graphs with a cyclic capture relation

Problem statement K players $(P_0, P_1, \ldots, P_{K-1})$ play a pursuit game on a finite, simple, undirected graph G. The game has the classic Cops and Robbers rules except for the use of a "Cyclic Capture Relationship": P_0 chases P_1, P_1 chases P_2 etc.. P_{K-1} chases P_0. More specifically, the rules are the following

(a) The starting positions (vertices) of the players are given.

(b) At the t-th turn the i-th player, where $i = (t \bmod K)$, has the move and the others stay in place.

(c) The player who has the move, can move to a vertex in the closed neighborhood of his current position.

(d) P_k wins (and the game ends) iff at the end of a turn he is in the same vertex as P_i, where $i = (k + 1 \bmod K)$. This is a "capture". If no capture takes place (at any turn) the game is a draw.

The basic is to "solve" the game, i.e., to establish the existence of a "reasonable" outcome and "reasonable" strategies for the players.

Additional remarks 1. To complete the description of the problem one must define an appropriate payoff function for each player. Many choices are possible. For example: if capture is effected by player P_k, he receives a payoff of 1 while all other players receive a payoff of -1. Other payoffs functions of a win/lose type can be used; furthermore payoffs may also depend on capture time.

2 Having defined a payoff function, the basic problem can be stated as follows: prove that the game possesses a Nash equilibrium (NE). The existence of a NE may depend on the cop number of the graph.

3. Additional problems include (but are not limited) to the following.

   (a) Prove the existence of Subgame Perfect Equilibrium (SPE).

   (b) Prove that every NE results in capture.

   (c) If (a) and/or (b) are not always true, Characterize graphs for which they hold.

   (d) Find an analog of the cop number.

Related work: A similar multiplayer pursuit game has been studied in [1, 2] Useful literature on multiplayer stochastic games includes [4, 3, 5].

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6 Euripides Markou: The Bumblebee Visitation problem

Statement of the problem Consider a connected undirected graph $G = (V, E)$, where each node $u \in V$ has been assigned a positive integer $c(u)$. A mobile agent A is able to move along the edges of the graph under the following rules. When agent A traverses an edge (u, v), the numbers $c(u)$ and $c(v)$ decrease by 1 (we say that A collects a value 1 from u and a value 1 from v). An edge (u, v) cannot be traversed if $c(u) = 0$ or $c(v) = 0$.

Design an algorithm which takes as input a graph G with positive values at nodes and places and moves an agent A as above, so that a maximum total value is collected.

Related work The problem (decision version) is NP-hard in arbitrary graphs due to a reduction from the Hamilton Circuit problem: assign a value 2 to each node of the given graph G; the graph has a Hamilton Circuit iff a total value of at least $2n$ can be collected, where n is the number of nodes of G.

Consider the following variation of the problem: Design an algorithm which takes as input a graph G with positive values at nodes and places and moves an agent A as above, so that the total value *remained at visited nodes* is a maximum. Some open questions are as follows:

- Study the above problems in tree or other topologies.
- Design approximation algorithms.
- Study distributed scenarios with more than one mobile agents that might communicate at a cost and have limited information about the topology (e.g., local maps) and/or limited memory, etc.

The above problems can have applications like the following one: Consider a network. The nodes of the network can host an application but each node has to spend some energy in order to execute the application. Initially each node u has an energy $c(u)$. The application can migrate in the network from a node u to a node v, if nodes u, v are adjacent nodes. For the migration procedure, each one of the nodes u, v spends an energy 1. Where to start the execution of the application and what is the migration tour on the network so that the application can use a maximum energy?

References Those problems and their applications were mentioned to me by my colleague assist. prof. Thanasis Loukopoulos. We do not know of any related results.

7 Sebastian Siebertz: Parameterized complexity of generalized coloring numbers

The *colouring number* $\mathrm{col}(G)$ of a graph G is the minimum integer k such that there is a strict linear order $<_L$ of the vertices of G for which each vertex v has *back-degree* at most $k-1$, i.e. at most $k-1$ neighbours u with $u <_L v$. It is well-known that for any graph G, the chromatic number $\chi(G)$ satisfies $\chi(G) \leq \mathrm{col}(G)$.

Some generalisations of the colouring number of a graph have been studied in the literature. Three natural generalisation of the colouring number are the three series adm_r, col_r and wcol_r of *generalised colouring numbers* introduced by Kierstead and Yang [7] in the context of colouring games and marking games on graphs. As proved by Zhu [7], these invariants are strongly related to low tree-depth decompositions [7], and can be used to characterise bounded expansion classes of graphs (introduced in [8]) and nowhere dense classes of graphs (introduced in [9]).

An interesting aspect of generalised colouring numbers is that these invariants can also be seen as gradations between the colouring number $\mathrm{col}(G)$ and two important minor monotone invariants, namely the *tree-width* and the *tree-depth*. More explicitly, for every graph G we have

$$\mathrm{col}(G) = \mathrm{col}_1(G) \leq \mathrm{col}_2(G) \leq \cdots \leq \mathrm{col}_\infty(G) = \mathrm{tw}(G) + 1 \quad \text{and}$$
$$\mathrm{col}(G) = \mathrm{wcol}_1(G) \leq \mathrm{wcol}_2(G) \leq \cdots \leq \mathrm{wcol}_\infty(G) = \mathrm{td}(G).$$

See [2] for a structure theorem for graphs with bounded adm_∞. While we can compute $\mathrm{col}(G)$ of a graph G in linear time, computing $\mathrm{col}_2(G)$ is NP-complete in general (this follows from Theorem 16 of [1]) and computing $\mathrm{wcol}_3(G)$ is NP-complete in general [4]. To the best of my knowledge it is open whether $\mathrm{wcol}_2(G)$ can be computed in polynomial time. Several approximations, e.g. on bounded tree-width graphs [4], on planar graphs and graphs with excluded minors [10] or on graphs with excluded topological minors [6] are known. It is known how to compute $\mathrm{adm}_r(G)$ in time $f(\mathrm{adm}_r(G)) \cdot n$ on an n-vertex graph G [3].

Question: Is it possible to compute $\mathrm{col}_r(G)$ and $\mathrm{wcol}_r(G)$ of an n-vertex graph in time $n^{f(\mathrm{col}_r(G))}$ and $n^{g(\mathrm{wcol}_r(G))}$, respectively, for computable functions f, g?

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