Cops, Robber and Medianwidth Parameters

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Crete

Tree Decompositions

A *tree decomposition* $\mathcal{D}$ of a graph G is a pair $(T, \mathcal{Z})$, where T is a *tree*, $\mathcal{Z} = (Z_t)_{t \in V(T)}$, is a family of subsets of $V(G)$ (called bags) such that

(T1) for every edge $uv \in E(G)$ there exists $t \in V(T)$ with $u, v \in Z_t$,

(T2) for every $v \in V(G)$, the set $Z^{-1}(v) := \{t \in V(T) \mid v \in Z_t\}$ is a non-empty *connected* subset (a subtree) of T.

Tree Decompositions

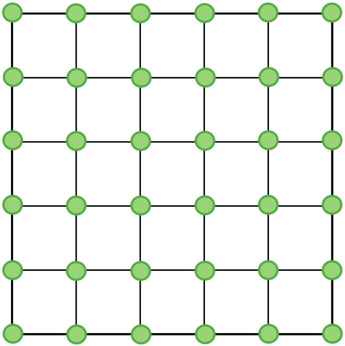
- The *width* of a tree decomposition $\mathcal{D} = (T, \mathcal{Z})$ is the number

$$\max\{|Z_t| - 1 \mid t \in V(T)\}.$$

- Let $\mathcal{T}^G$ be the set of all tree decompositions of G. The *treewidth* $\mathrm{tw}(G)$ of G is

$$\mathrm{tw}(G) := \min_{\mathcal{D} \in \mathcal{T}^G} \max\{|Z_t| - 1 \mid t \in V(T)\}.$$

The Grid



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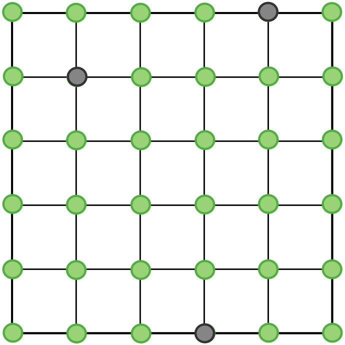

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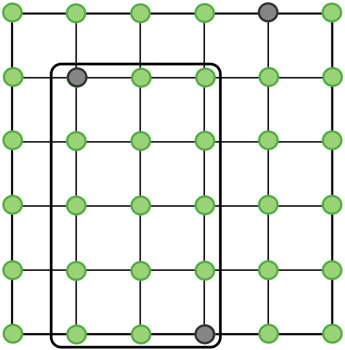

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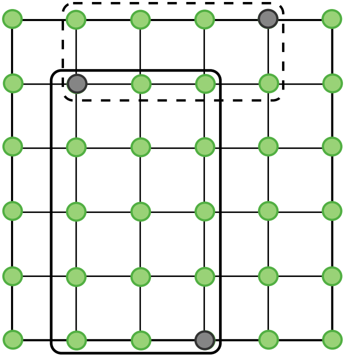

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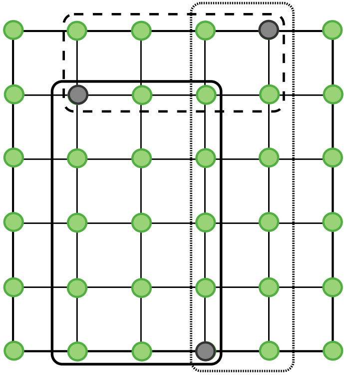

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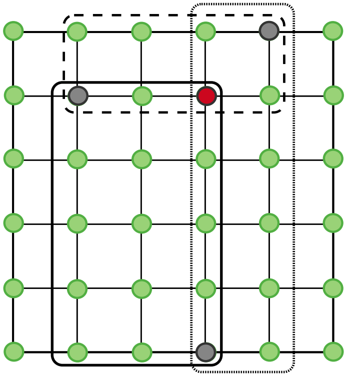


Median Graphs

- *Interval* $I(u,v)$: all vertices on a shortest (u,v)-path (a (u,v)-geodesic):

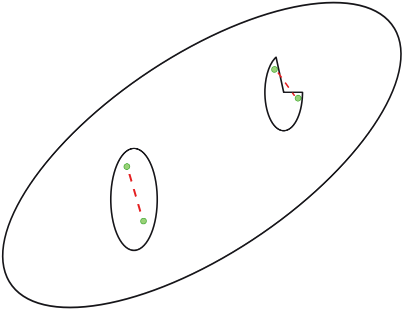
$$I(u,v) = \{x \in V(G) \mid d(u,v) = d(u,x) + d(x,v)\}.$$

- *Median Graph*: connected and for any vertices u,v,w there is a unique vertex x (the *median* of u,v,w) lying on a shortest (u,v)-path, a shortest (v,w)-path and a shortest (w,u)-path:

$$|I(u,v) \cap I(v,w) \cap I(w,u)| = 1$$

Convexity

A vertex set S is a *convex subset* of a graph if for every $u, v \in S$, $I(u, v) \subseteq S$.



Not Enough Space

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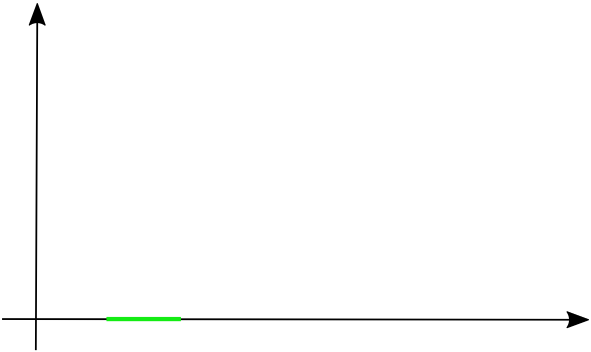

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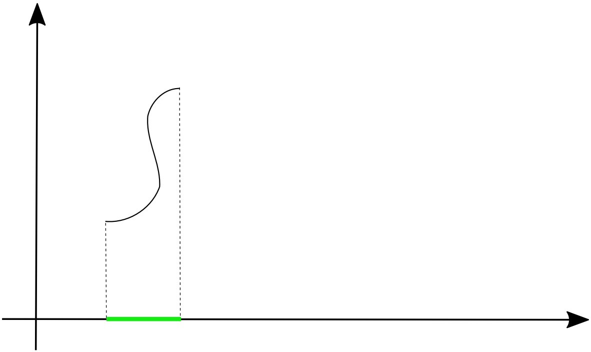

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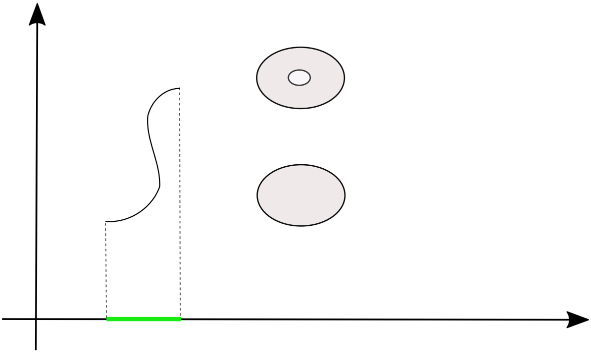

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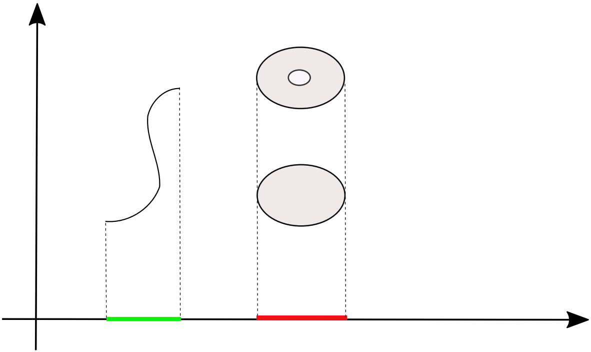

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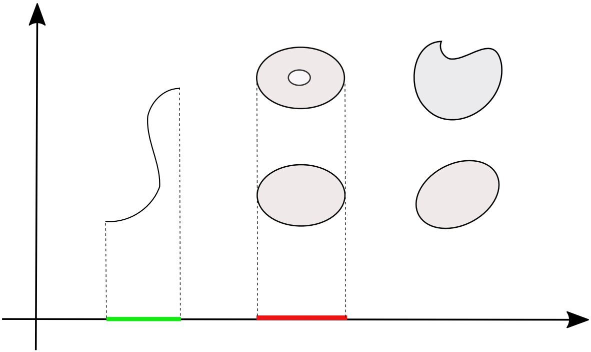

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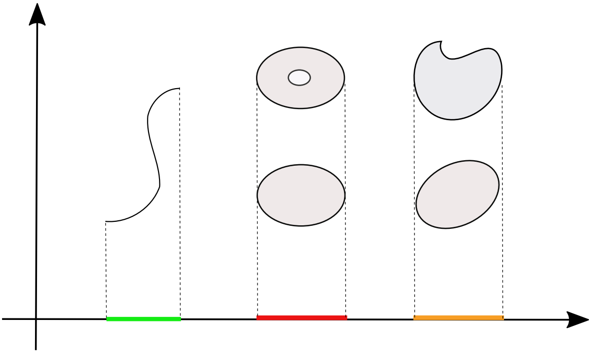

The Crucial Observation

Convexity degenerates to connectedness on trees!

Tree Decompositions

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Median Decompositions

A *median decomposition* $\mathcal{D}$ of a graph G is a pair $(M, \mathcal{X})$, where M is a *median graph*, $\mathcal{X} = (X_a)_{a \in V(M)}$, is a family of subsets of $V(G)$ (called bags) such that

(M1) for every edge $uv \in E(G)$ there exists $a \in V(M)$ with $u, v \in X_a$,

(M2) for every $v \in V(G)$, the set $X^{-1}(v) := \{a \in V(M) \mid v \in X_a\}$ is a non-empty *convex* subset of M.

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- Let $\mathcal{M}^G$ be the set of all median decompositions of G. The *medianwidth* $\mathrm{mw}(G)$ of G is
$$\mathrm{mw}(G) := \min_{\mathcal{D} \in \mathcal{M}^G} \max\{|X_a| \mid a \in V(M)\}.$$

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$$\mathrm{mw}(G) := \min_{\mathcal{D} \in \mathcal{M}^G} \max\{|X_a| \mid a \in V(M)\}.$$
- By definition, $\mathrm{mw}(G) \leq \mathrm{tw}(G) + 1$.

The Clique Number

Theorem

For any graph G, $\mathrm{mw}(G) = \omega(G)$.

Bounding the Dimension

- *tree dimension*: the minimum k such that G has an isometric embedding into a Cartesian product of k trees.
- *lattice dimension*: the minimum k such that G has an isometric embedding into a Cartesian product of k paths.

- Median graphs have finite lattice and tree dimension.

Bounding the Dimension

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- $\mathcal{M}_i^G$: the set of i-median decompositions of G
- The *i-medianwidth* $\mathrm{mw}_i(G)$ of G is

$$\mathrm{mw}_i(G) := \min_{\mathcal{D}\in\mathcal{M}_i^G} \max\{|X_a| \mid a \in V(M)\}.$$

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- An *i-lattice decomposition* of G is a median decomposition $\mathcal{D} = (M, \mathcal{X})$ satisfying (M1),(M2), where M has lattice dimension at most i
- $\mathcal{L}_i^G$: the set of i-lattice decompositions of G
- The *i-latticewidth* $\mathrm{lw}_i(G)$ of G is

$$\mathrm{lw}_i(G) := \min_{\mathcal{D} \in \mathcal{L}_i^G} \max\{|X_a| \mid a \in V(M)\}.$$

Bounding the Dimension

- $\mathrm{mw}_1(G) = \mathrm{tw}(G) + 1$, $\mathrm{lw}_1(G) = \mathrm{pw}(G) + 1$

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- $\mathrm{mw}_1(G) = \mathrm{tw}(G) + 1$, $\mathrm{lw}_1(G) = \mathrm{pw}(G) + 1$
- the invariants $\mathrm{mw}_i, \mathrm{lw}_i$ form non-increasing sequences:

$$\mathrm{tw}(G) + 1 = \mathrm{mw}_1(G) \geq \mathrm{mw}_2(G) \geq \cdots \geq \mathrm{mw}_\infty(G) = \omega(G).$$

$$\mathrm{pw}(G) + 1 = \mathrm{lw}_1(G) \geq \mathrm{lw}_2(G) \geq \cdots \geq \mathrm{lw}_\infty(G) = \omega(G).$$

i-Medianwidth

Theorem

For any graph G and any integer $i \geq 1$,

$$\mathrm{mw}_i(G) = \min_{\mathcal{D}^1,\ldots,\mathcal{D}^i \in \mathcal{T}^G} \max\{|\bigcap_{j=1}^{i} Z^j_{t_j}| \mid t_j \in V(T^j)\}.$$

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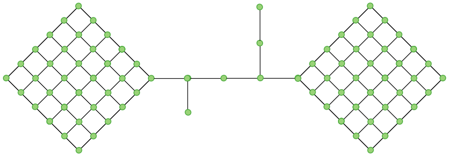
i-Medianwidth

Intuition



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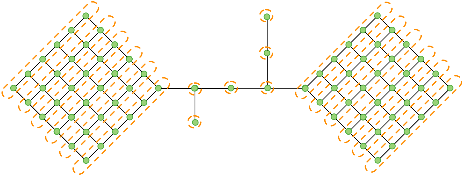

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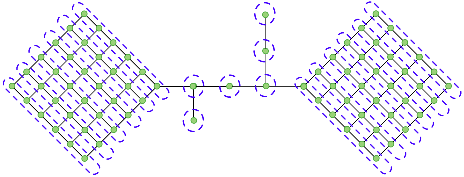

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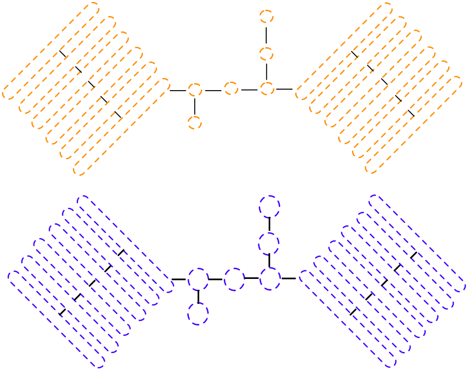

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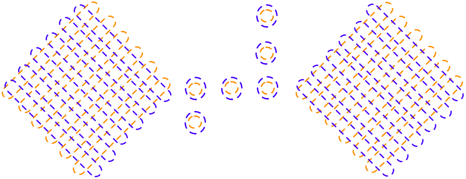
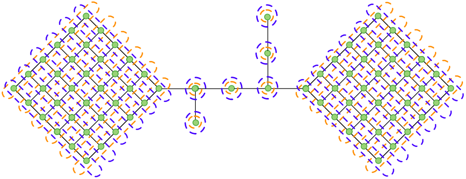


i-Medianwidth

Proof Intuition



A Game Characterisation

Cops and Robber

- A team of k Cops try to capture a Robber on a graph G, robber wants to escape for ever.
- In each round: some cops move with a helicopter from one vertex to another
- while the cops are in the air, the Robber can arbitrarily fast move to any vertex through a path in the graph...
- ...as long as there is no Cop on the ground standing on a vertex in the path.

A Game Characterisation

Two variations

- (Seymour and Thomas) A team of k Cops can always capture a *visible* Robber on a graph G if and only if $\mathrm{tw}(G) \leq k - 1$.
- (Lapaugh, Kirousis and Papadimitriou, Bienstock et al) A team of k Cops can always capture an *invisible* Robber on a graph G if and only if $\mathrm{pw}(G) \leq k - 1$.

A Game Characterisation

i-Cops and Robber

- i different teams of $|V(G)|$ cops each try to capture a Robber simultaneously on a graph G.
- cop teams "invisible" to each other
- need to cooperate in order to capture the robber with the least "cooperation" possible
- robber is powerful: she can be fully caught only if every team captures her at the same time.
- she chooses which team of cops can move in each round

A Game Characterisation

i-Cops and Robber

- cops: when a team of cops is chosen, they play as in the classical game
- cops also have some power; robber still can't travel through them
- if a team captures the Robber, then it locks down and can't be chosen to move again for the rest of the game; the robber from then on can only move to vertices occupied by the team.
- cop *cooperation*: the maximum number of vertices simultaneously occupied by a cop of every team at any point of the game.

A Game Characterisation

Two variations

- i cop teams can *monotonely* capture a *visible* robber on a graph G with cooperation at most k if and only if $\mathrm{mw}_i(G) \leq k$.
- i cop teams can *monotonely* capture an *invisible* robber on a graph G with cooperation at most k if and only if $\mathrm{lw}_i(G) \leq k$.

Non-monotonicity in i-Cops and Robber game

- For treewidth, monotone and non-monotone strategies are equally strong.

Non-monotonicity in i-Cops and Robber game

- For treewidth, monotone and non-monotone strategies are equally strong.
- What is the case for i-medianwidth?

Thank you for your attention!