Cops and Robbers with Gangs (on Graphs of Small Girth)

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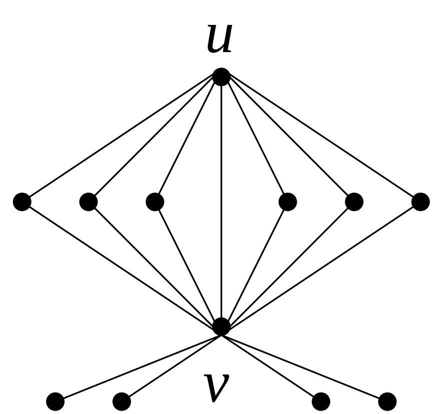
Cops and Robber

Rules of the Game

- two opposing sides, $k > 0$ cops and a single robber
- both sides play with perfect information
- cops begin the game by each choosing a vertex to occupy then robber chooses a vertex
- opposing sides move alternately
- cops win if at least one of them occupies the same vertex as the robber after a finite number of moves

A subgraph H of a graph G is a **retract** of G if there is a homomorphism $f : V(G) \to V(H)$ such that $f(x) = x$, for all $x \in V(H)$.

A vertex u of a graph G is **irreducible** or a **corner** if there exists a vertex v in G such that $N[u] \subseteq N[v]$. We say that the vertex v **dominates** u.

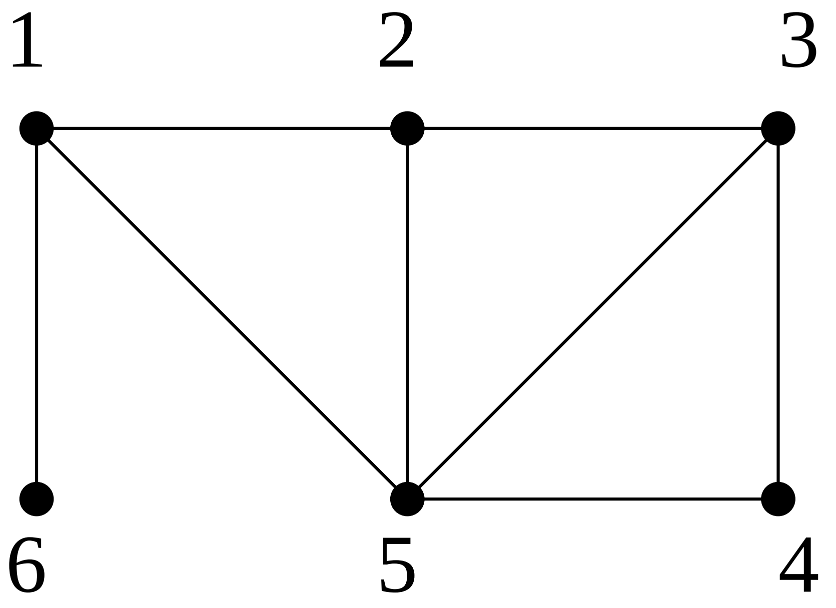


Define $G_i = G - \{v_1, v_2, \ldots, v_{i-1}\}$.

A vertex ordering $(v_1, v_2, \ldots, v_n)$ on a graph G is a **domination elimination ordering** if, for all $i \in \{1, 2, \ldots, n-1\}$, there is a $j_i > i$ such that $N_i(v_i) \subseteq N_i[v_{j_i}]$ in G_i.

If, in addition, $v_i \sim v_{j_i}$, then this domination elimination ordering is a **copwin ordering**.

Theorem (Nowakowski & Winkler) *A finite graph G is copwin if and only if G has a copwin ordering.*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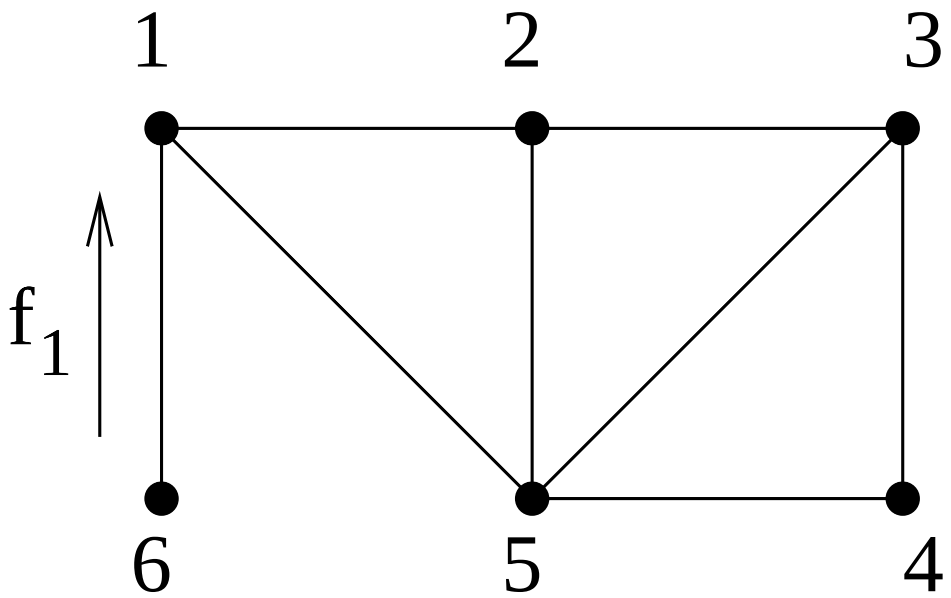
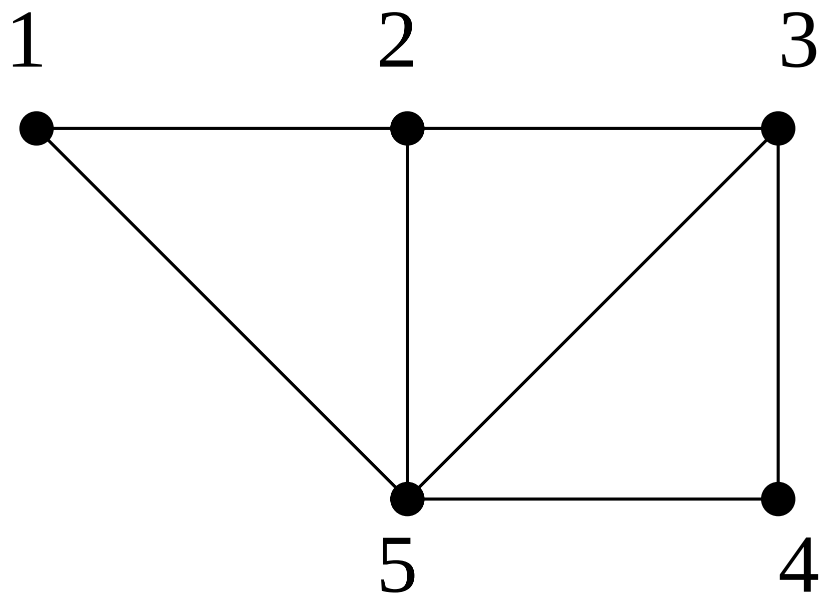


copwin ordering



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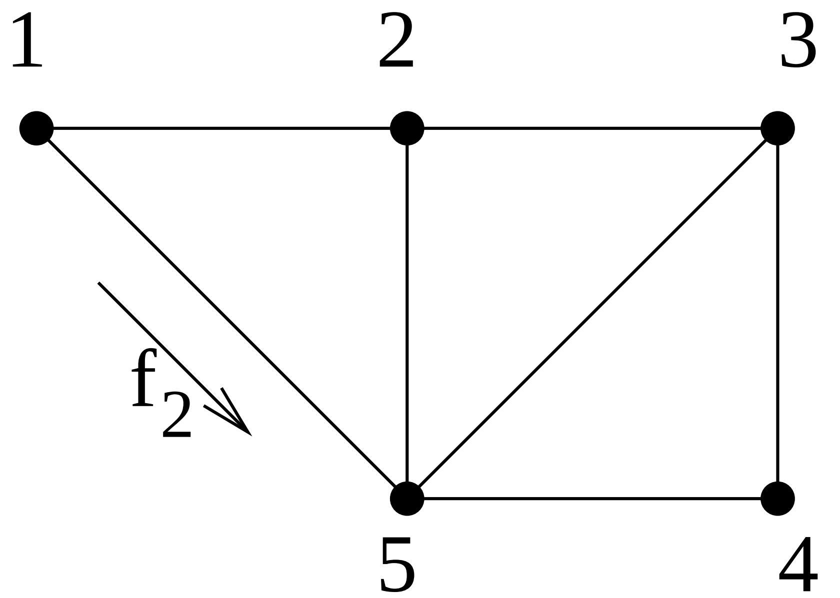

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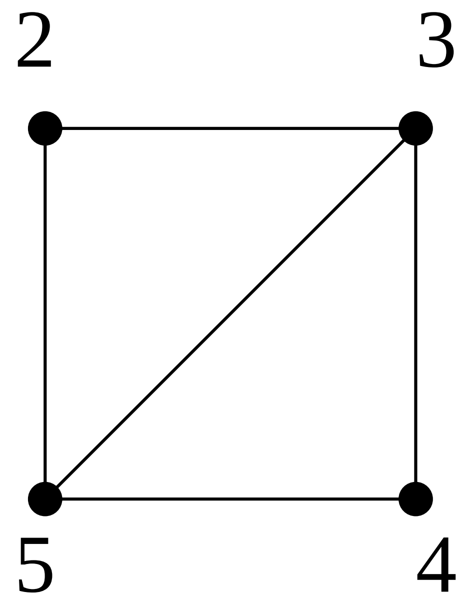

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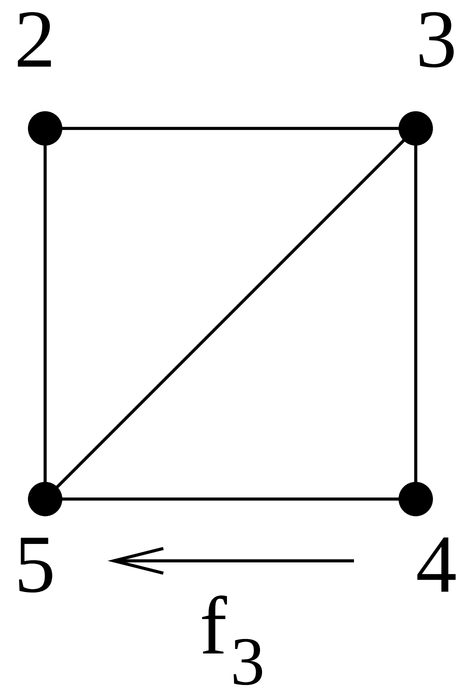

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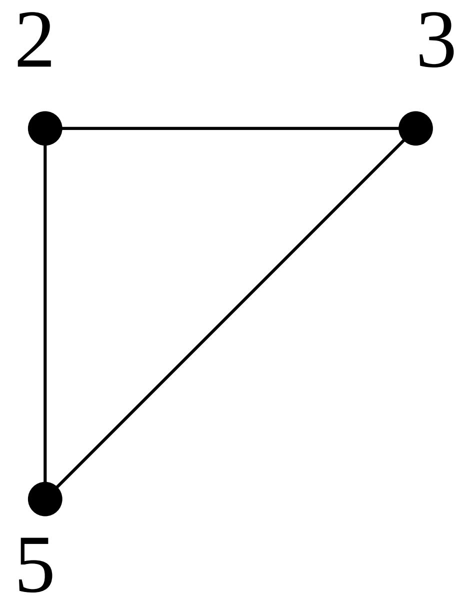

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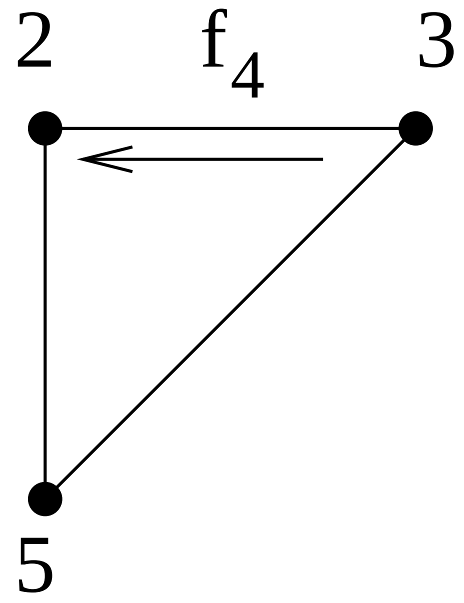

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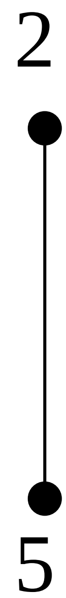

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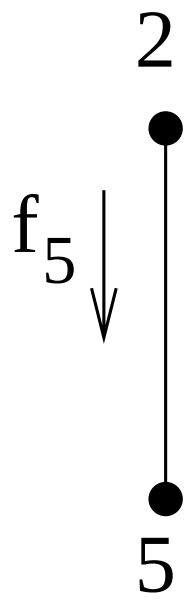

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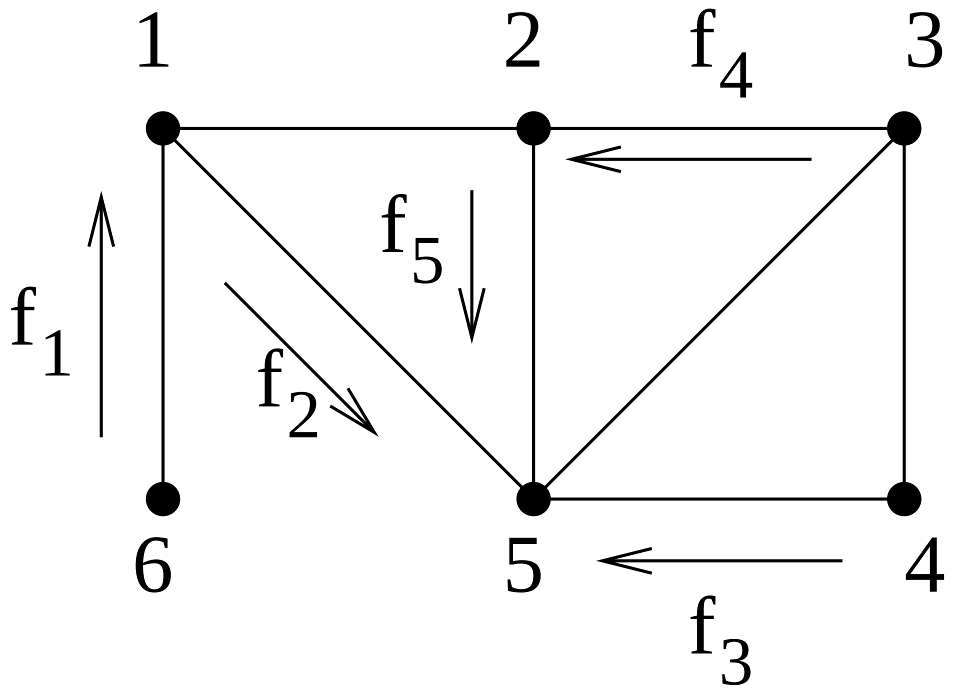
copwin ordering (6, 1, 4, 3, 2

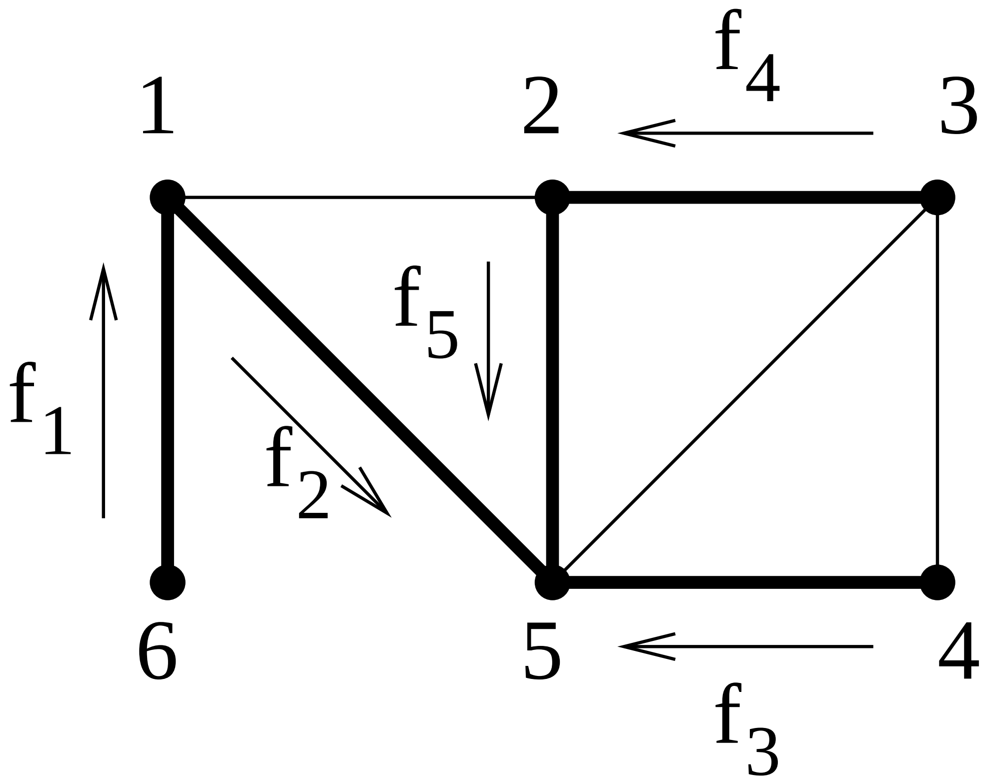


copwin ordering (6, 1, 4, 3, 2, 5) .

Consider a fixed copwin ordering $(x_1, x_2, \ldots, x_n)$ of G.

The corresponding **copwin spanning tree** is the spanning tree S on $V(G)$, rooted at x_n, with the property that for $x_1, x_2 \in V(G)$, $x_1x_2 \in E(S)$ if and only if $f_j(x_1) = x_2$ (denoted $x_1 \to x_2$) or $f_j(x_2) = x_1$, for some j.

We say that $x_1 \preceq x_2$ if x_1 is eventually retracted onto x_2 and $x_1 \prec x_2$ if $x_1 \neq x_2$.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Copwin Strategy

Let $(x_1, x_2, \ldots, x_n)$ be a copwin ordering of the vertices of G and let f_j be a retraction of G_j to G_{j+1}.

If the robber is on vertex x, define

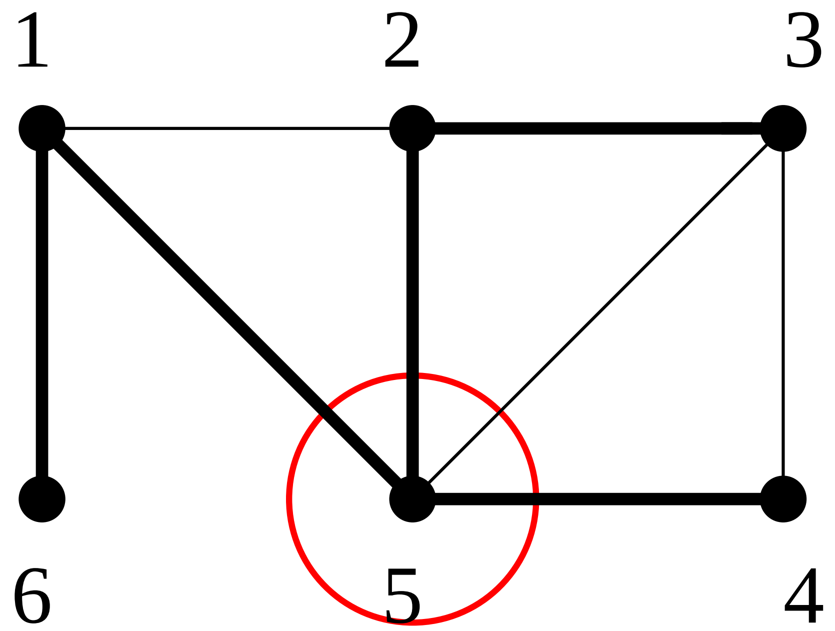
$$F_i(x) = f_{i-1} \circ f_{i-2} \circ \cdots \circ f_2 \circ f_1(x)$$

so that $F_i(x)$ is the robber's image on G_i.

The cop begins on vertex x_n.

If the cop has captured F_i on the subgraph G_i, he is able to move so as to immediately capture F_{i-1} on G_{i-1}.

Theorem *The Copwin Strategy is effective in capturing the robber.*



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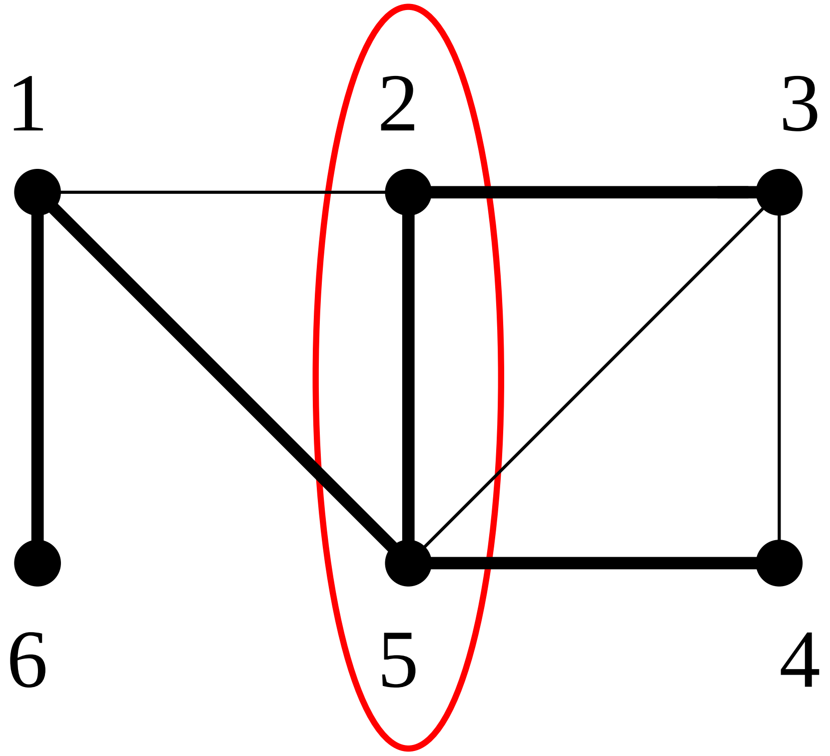

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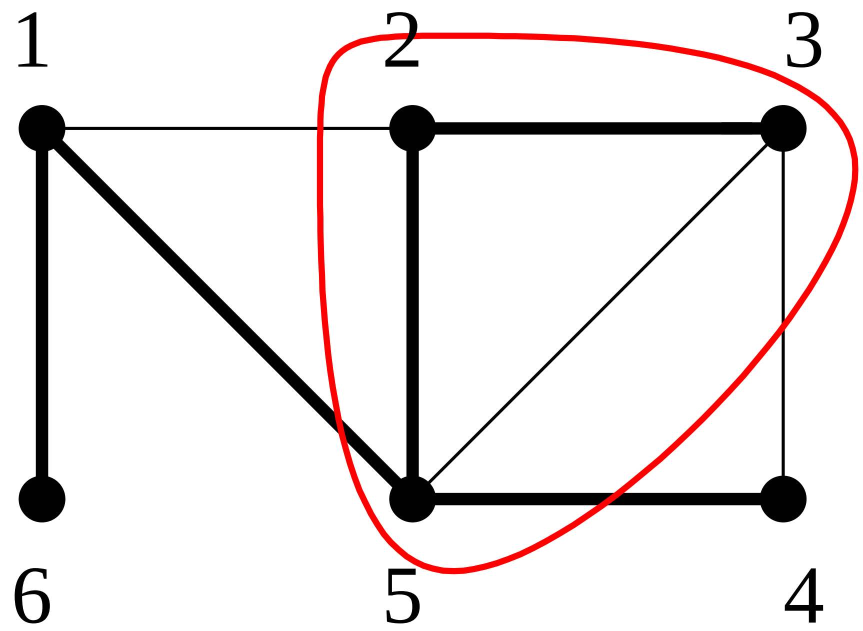

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The cop begins on vertex x_n.

If the cop has captured F_i on G_i, he may assume that he is playing on G_j, $i \geq j \leftarrow$ smallest index such that $F_j(x) = F_i(x)$. He is then able to move so as to immediately capture F_{j-1} on G_{j-1}.

Theorem *The Copwin Strategy is effective in capturing the robber.*

Rules of the C&R Game with Gangs

- two opposing sides, $k > 0$ cops and two robbers
- both sides play with perfect information
- cops begin the game by each choosing a vertex to occupy then robbers choose distinct vertices
- opposing sides move alternately
- cops win if at least one of them occupies the same vertex as one of the robbers after a finite number of moves i.e. robbers win if they can avoid capture indefinitely
- robbers also win by simultaneously moving onto the same vertex as a cop
- otherwise the robbers must be located on distinct vertices

Some Easy (But Interesting!) Observations

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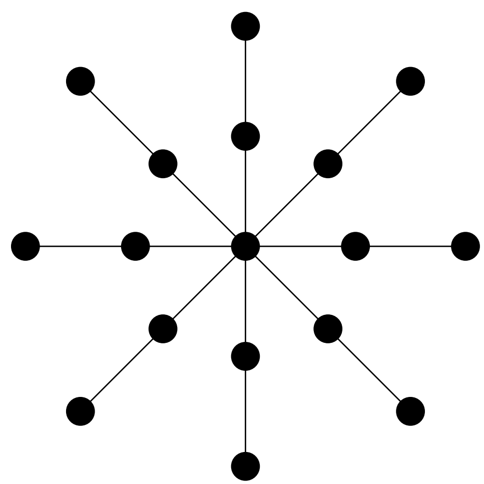

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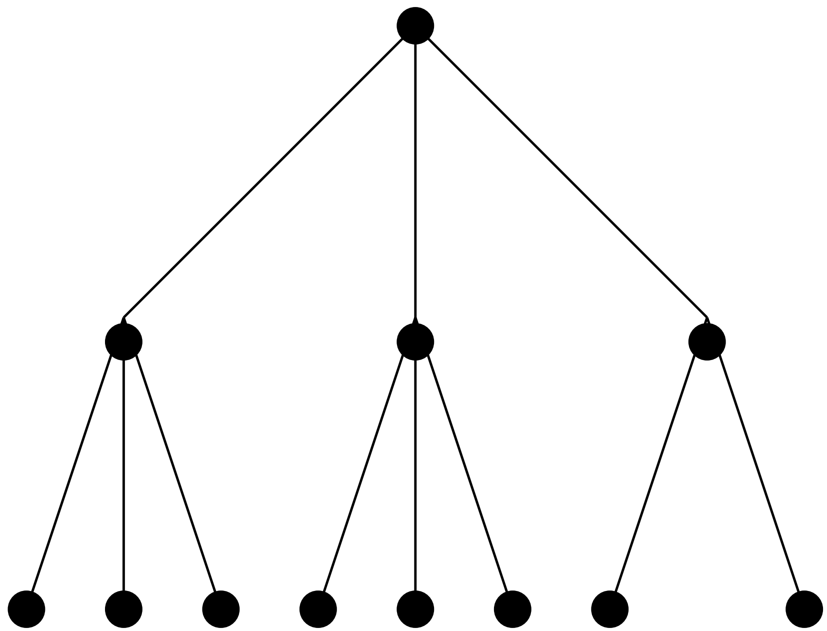

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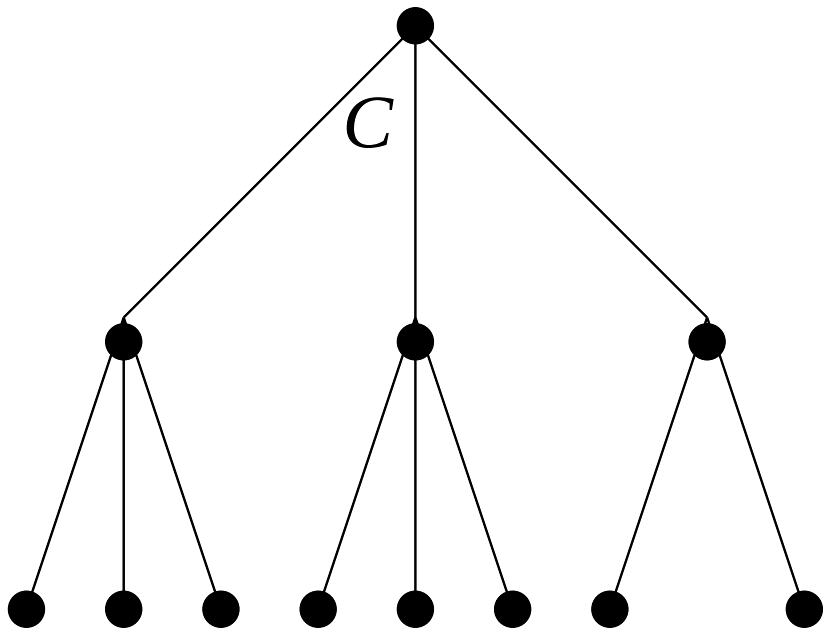

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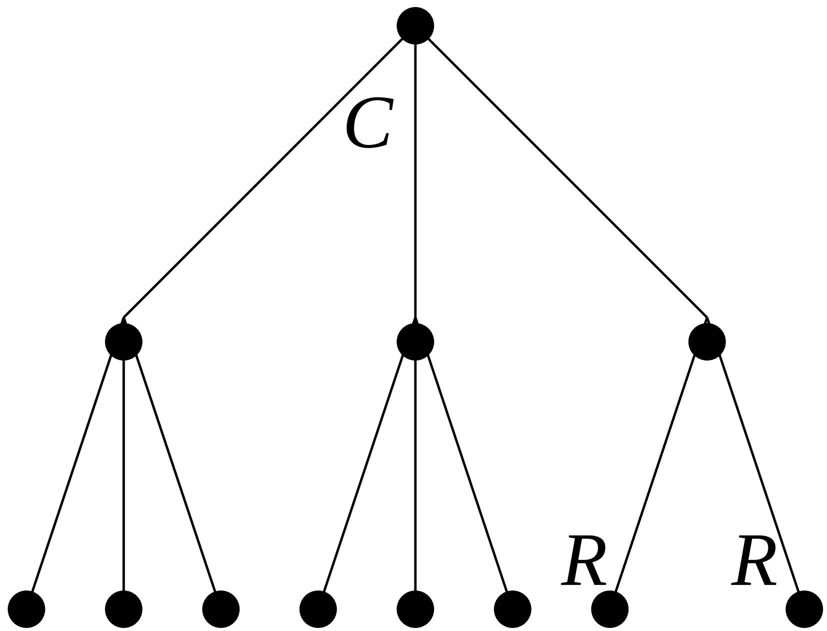

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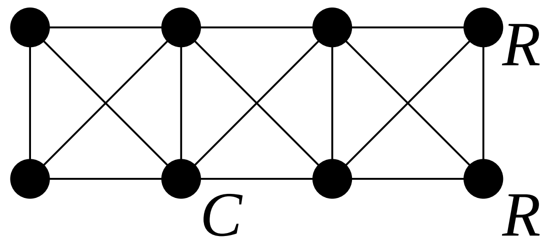

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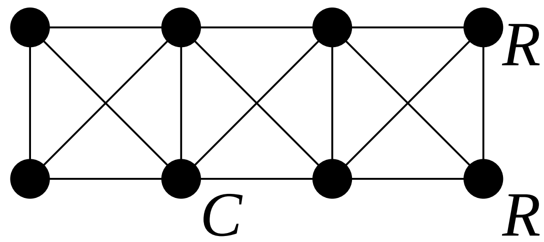
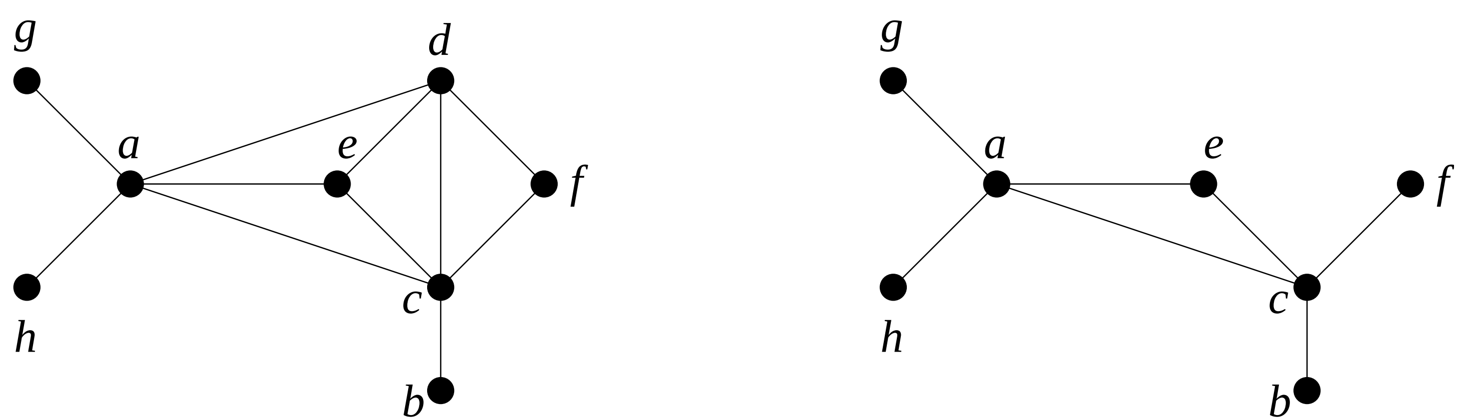


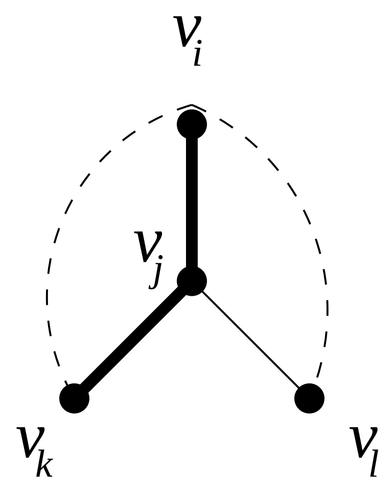


(a) (b)

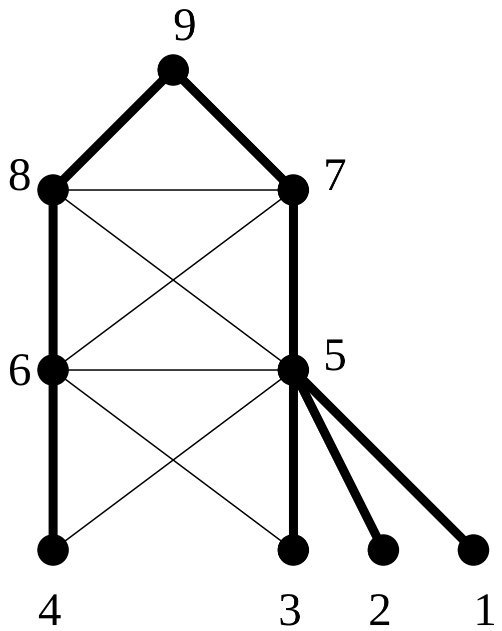
When can the cop use the Copwin Strategy?

Theorem Let G be a copwin graph. If there exists a spanning tree $\mathcal{T}$ of G such that some collection of forbidden subgraphs depending on $\mathcal{T}$ are not contained in G, then G is gang copwin.

Definition Let G be a graph with copwin ordering $(v_1, v_2, \ldots, v_n)$ and corresponding copwin spanning tree $\mathcal{T}$. If $K_{1,3}$ with vertices v_i, v_j, v_k, and v_l is a subgraph of G such that $v_k \to v_j$, $v_j \to v_i$, and $v_i \perp v_k, v_i \perp v_l$, we denote this subgraph $S_{\mathcal{T}}$.



Example



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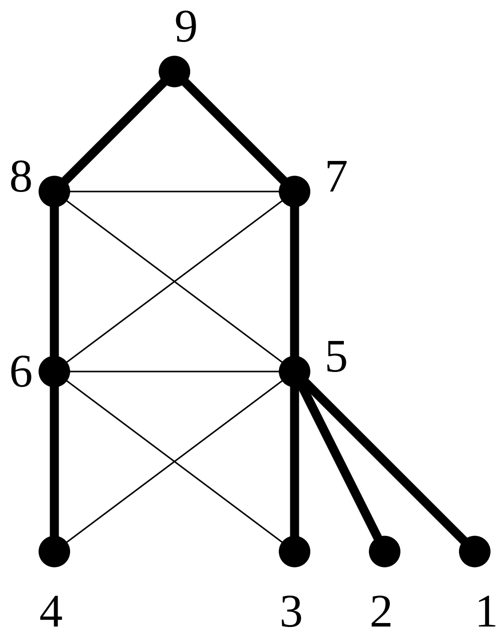
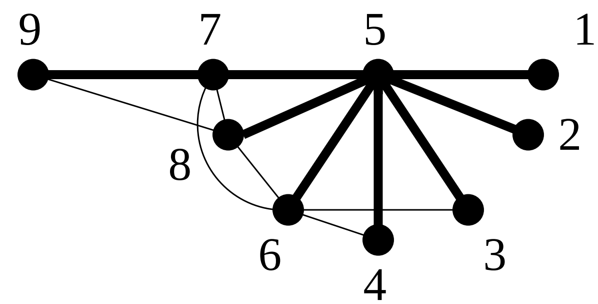
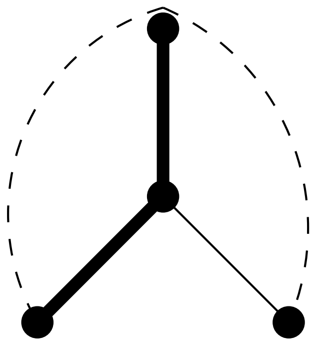




Can we do better?



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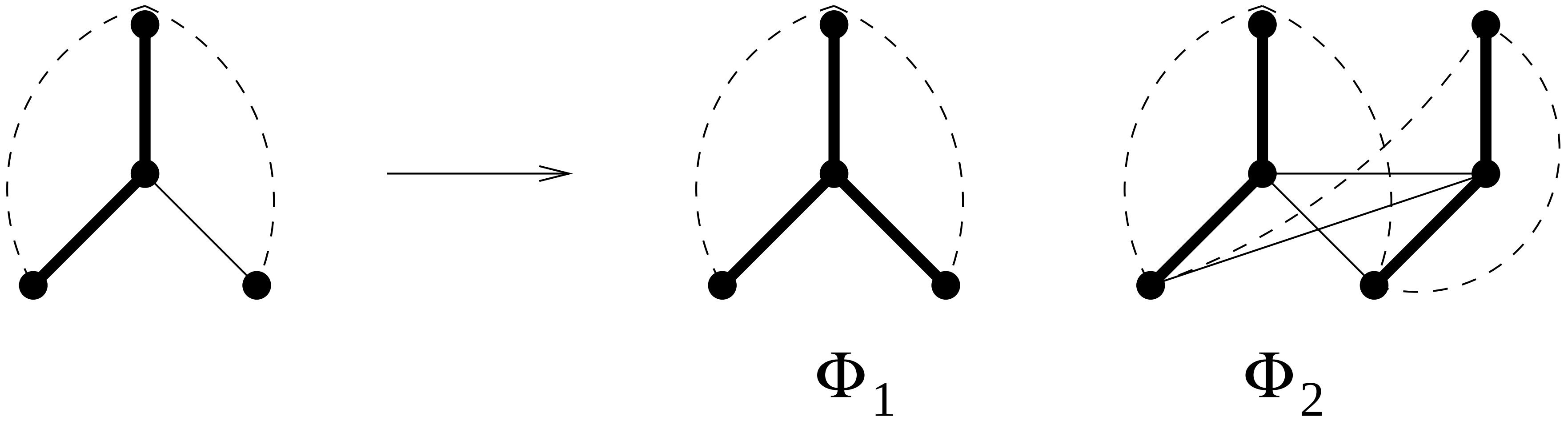

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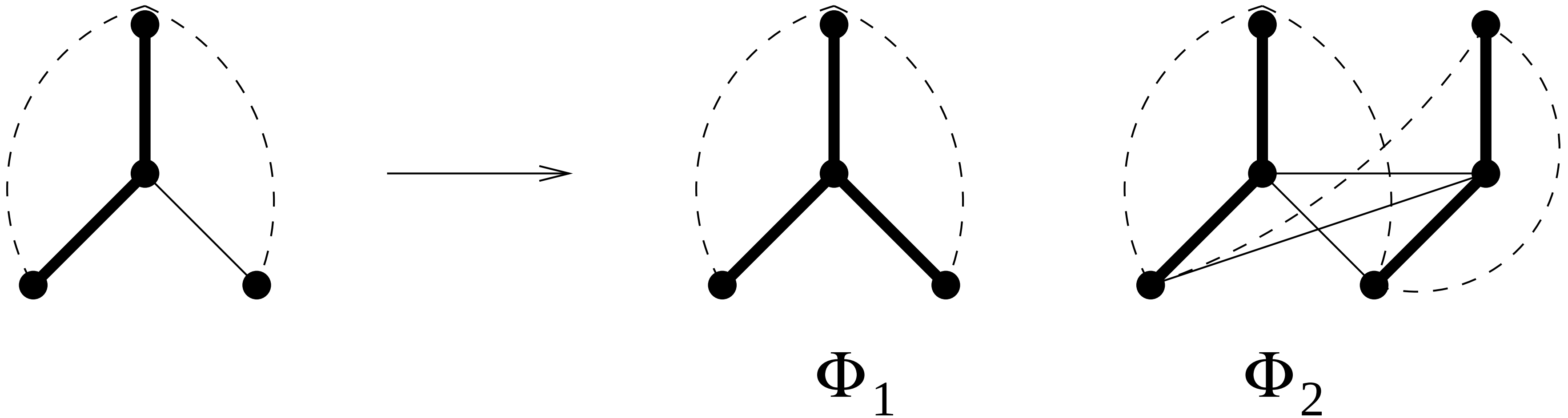
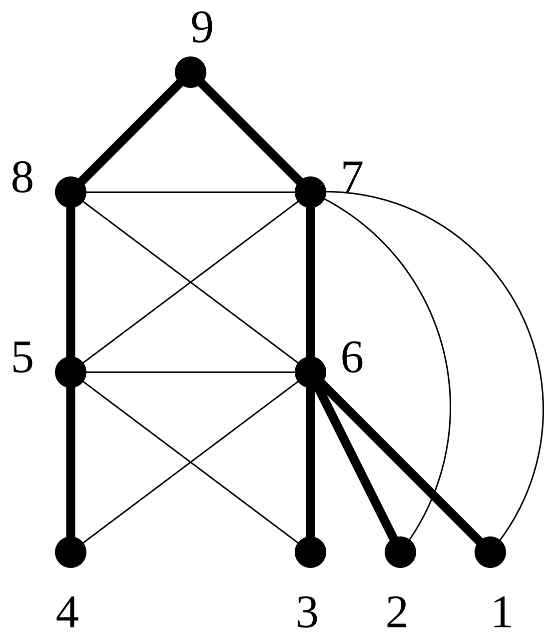


Forbidden Subgraphs

But we have to be careful!



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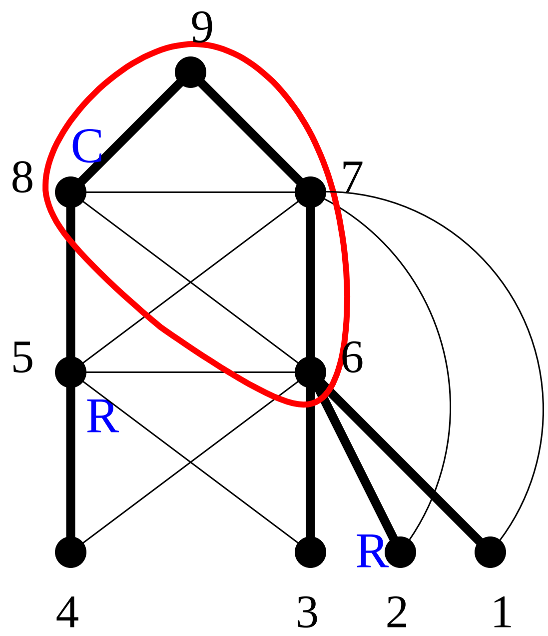

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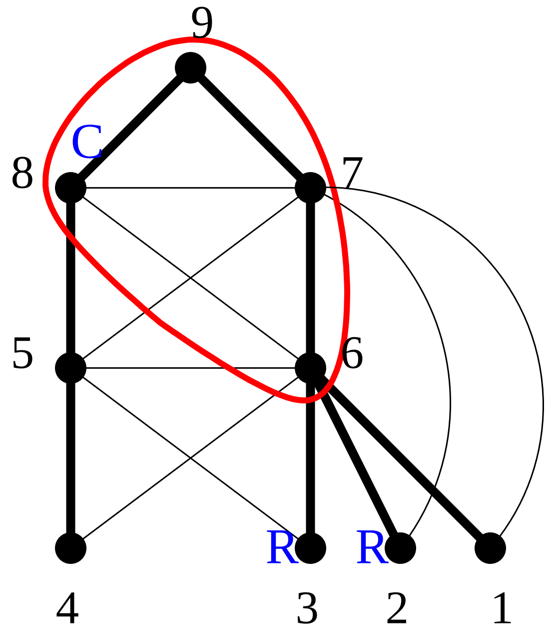

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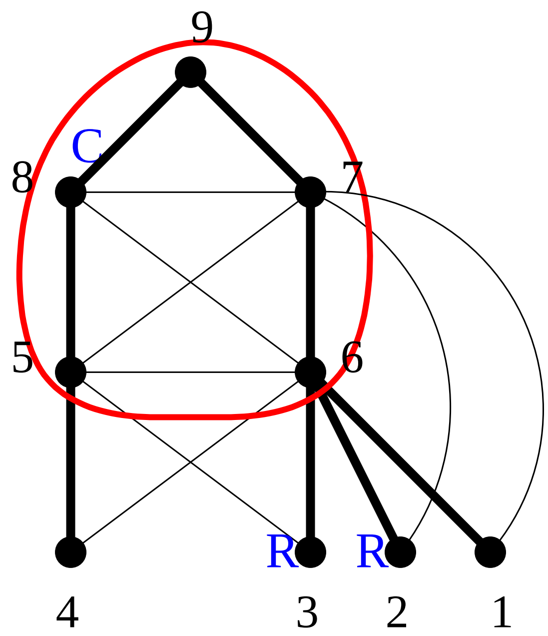

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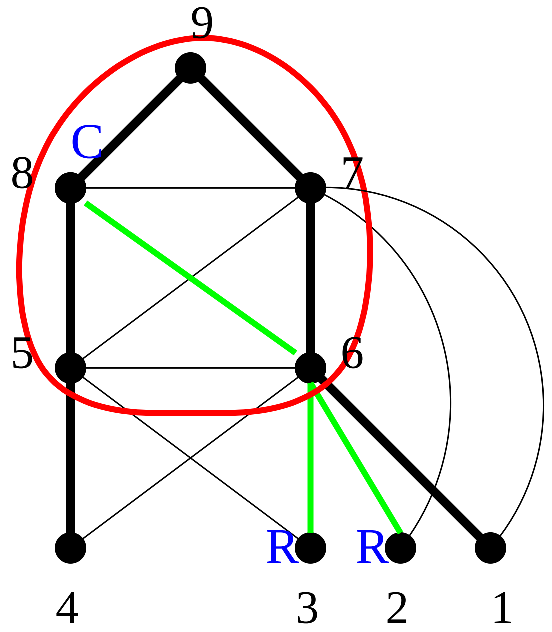

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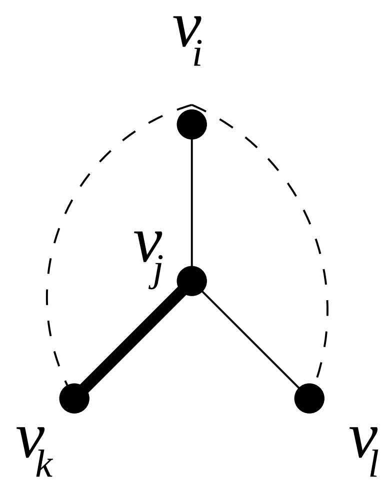

Structural Results

Lemma A gang-copwin graph G has a corner or three vertices a, b and b' such that

$$(N[b] \setminus \{b'\}) \cup N(b') \subseteq N[a].$$

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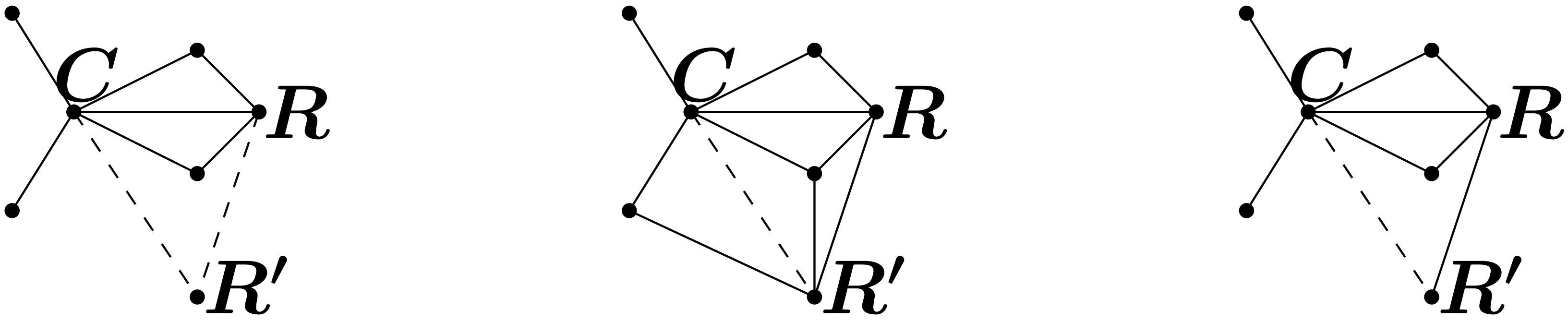

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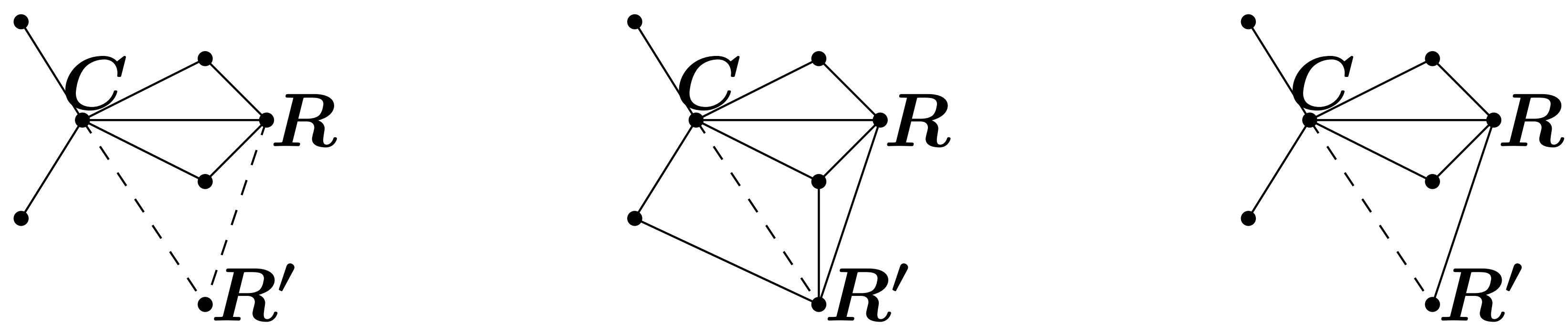


(a) (b) (c)

Proof shows that gang-copwin graphs must be generalized windmills or else have 3– or 4–cycles.

Structural Results

Theorem If G is gang-copwin and $u \in V(G)$ is simplicial, then $H = G - u$ is gang-copwin.

Girth 4

Theorem A triangle-free graph G is gang-copwin if and only if $G \cong K_{2,m}$ or $G \cong K_{2,m} \cup W$, $m \geq 2$, where W is a windmill such that its pivot p coincides with an m-vertex of $K_{2,m}$.

Girth 3

Theorem Let G be a gang-copwin graph with no adjacent triangles, $g(G) = 3$, and $\delta(G) \geq 2$. Then
(i) the intersection of all the triangles in G is a single vertex x;
(ii) if $G \ncong D_3^n$, then $G_n \cong K_{2,m}$, for some $m \geq 2$;
(iii) successive simplicial eliminations in G result in a single vertex if and only if $G \cong D_3^n$, $n \geq 1$;
(iv) G is either D_3^n or $D_3^n \cup K_{2,m}$ such that the pivot of the Dutch windmill coincides with an m-vertex of $K_{2,m}$ and some of its triangles may share an edge with $K_{2,m}$.

Girth 3

Let $\boldsymbol{W}$ be a windmill and let $\lhd_1 \boldsymbol{K_{2,m}}$ & $\lhd_2 \boldsymbol{K_{2,m}}$ denote $\boldsymbol{K_{2,m}} \cup \boldsymbol{K_3}$, $\boldsymbol{m \geq 2}$, where $\boldsymbol{K_3}$ shares one & two vertices with $\boldsymbol{K_{2,m}}$, respectively. If $\boldsymbol{a}$ is the triangular $\boldsymbol{m}$-vertex & $\boldsymbol{x}$ is simplicial, then

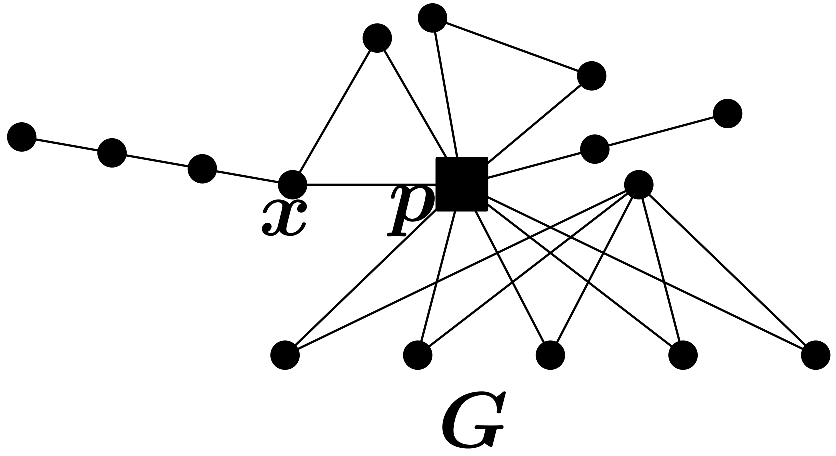
Theorem The following graphs are the only gang-copwin graphs with no intersecting triangles:

(i) $\triangleleft_2 K_{2,2} \cup W$, where the pivot of W coincides with x,

(ii) $\triangleleft_2 K_{2,m} \cup W \cup P$ and its subgraphs obtained by successive simplicial eliminations, where W has its pivot at a and the path P has x as one of its end vertices,

(iii) $\triangleleft_1 K_{2,m} \cup W \cup P_1 \cup P_2 \cup P_3$ and its subgraphs obtained by successive simplicial eliminations, where W has its pivot at a, P_1 starts at the other m-vertex, and P_2 and P_3 start on 2-vertices of the triangle, and

(iv) $K_3 \cup W \cup P_1 \cup P_2$, where the pivot of W and an end vertex of each of P_1 and P_2 are at distinct vertices of K_3.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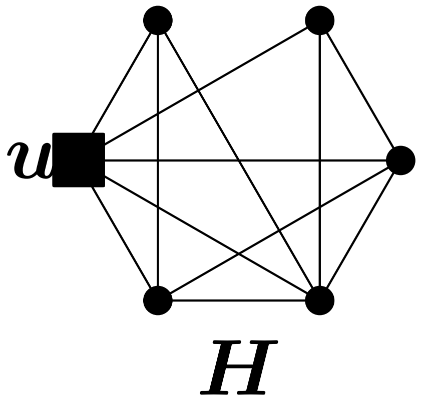
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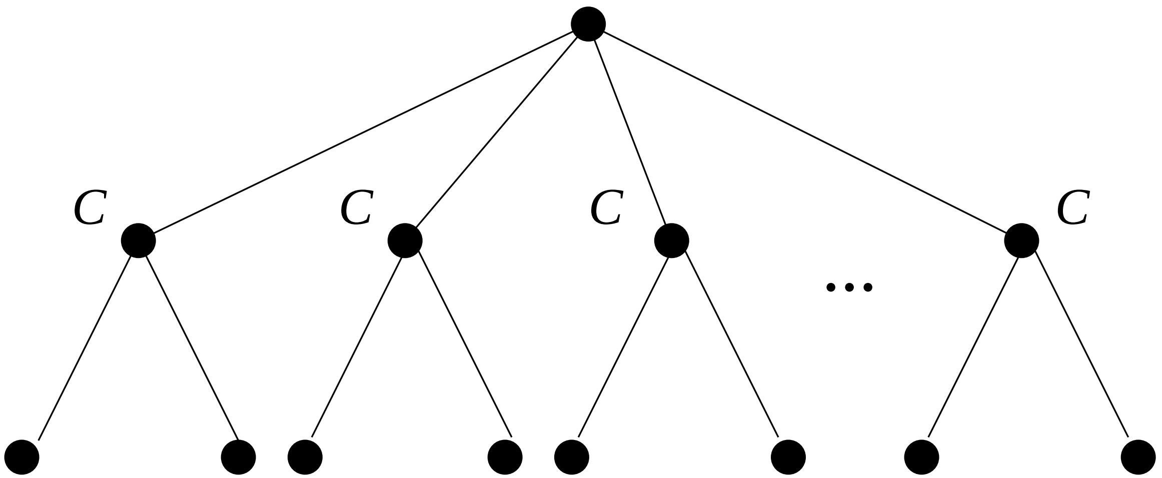
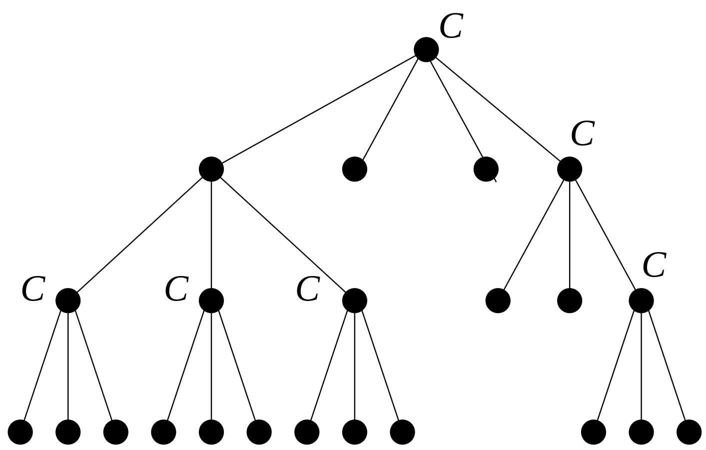
Copwin graph with arbitrarily high gang-copnumber



But we should think about the rules!



What if the robbers can't "gang up" on a vertex with 2 or more cops?

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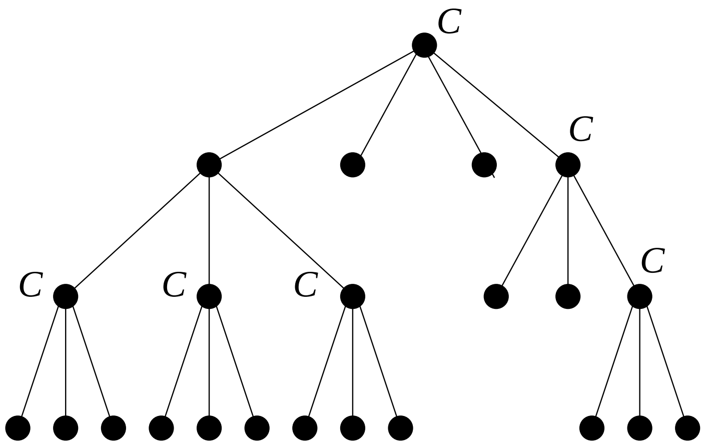

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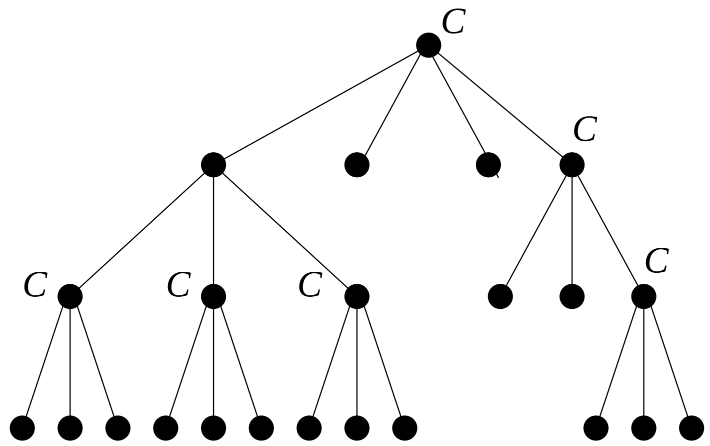

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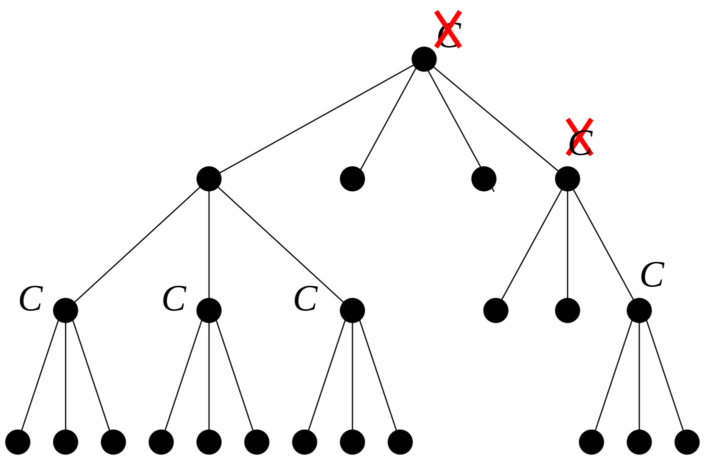

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